April 12 Math 3260 sec. 51 Spring 2024

Section 5.3: Diagonalization

Definition:

Two $n \times n$ matrices A and B are said to be **similar** if there exists an invertible matrix P such that

$$B = P^{-1}AP.$$

The mapping $A \mapsto P^{-1}AP$ is called a **similarity transformation.**

- If A and B are similar, they have the same characteristic equation and hence the same eigenvalues.
- If $B = P^{-1}AP$ for $n \times n$ matrices A, B, and P, then $B^k = P^{-1}A^kP$ for each integer $k \geq 1$.

Diagonalizability

Defintion:

An $n \times n$ matrix A is called **diagonalizable** if it is similar to a diagonal matrix D. That is, provided there exists a nonsingular matrix P such that $D = P^{-1}AP$—i.e. $A = PDP^{-1}$.

Theorem:

The $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In this case, the matrix P is the matrix whose columns are the n linearly independent eigenvectors of A.

Example

Diagonalize the matrix A if possible. $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

We found that the characteristic polynomial for A was $(1-\lambda)(\lambda+2)^2$ giving two distinct eigenvalues,

$$\lambda_1 = 1, \quad \text{and} \quad \lambda_2 = \lambda_3 = -2.$$

We found three linearly independent eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Then $D = P^{-1}AP$ where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Example

Diagonalize the matrix A if possible. $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. (With a little effort, it can be shown that the characteristic polynomial of A is $(1-\lambda)(2+\lambda)^2$.)

From the characteristic poly, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = -2$.

Find eigenvectors for $\lambda_1 = 1$.

Solve $(A - 1I)\vec{x} = \vec{0}$.

$$A - 1I = \begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_3$

$x_2 = -x_3$

x_3 free

$\vec{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ Let $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Find eigenvectors for $\lambda_2 = \lambda_3 = -2$

$$A - (-2)I = \begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_2$
x_2 - free
$x_3 = 0$

$x_1 + x_2 = 0$
$x_3 = 0$

$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

There are only two linearly independent eigenvectors.

So A is not diagonalizable.

Sufficient Condition for Diagonalizability

Recall: Eigenvectors corresponding to different eigenvalues are linearly independent.

Theorem:

If the $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Remark

This is a *sufficiency* condition, not a *necessity* condition. This means that if a matrix has n different eigenvalues, it's guaranteed to be diagonalizable. If it has repeated eigenvalues, it may or may not be diagonalizable.

More on Diagonalizability

Theorem:

Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$.

(a) The geometric multiplicity of λ_k is less than or equal to the algebraic multiplicity of λ_k.

(b) The matrix is diagonalizable if and only if the sum of the geometric multiplicities is n.

(c) If A is diagonalizable, and $\mathcal{B}_k$ is a basis for the eigenspace for λ_k, then the collection (union) of bases $\mathcal{B}_1, \ldots, \mathcal{B}_p$ is a basis for $\mathbb{R}^n$.

Remark: The union of the bases referred to in part (c) is called an **eigenvector basis** of $\mathbb{R}^n$ for the matrix A. This is what was called an **eigenbasis** in the 3Blue 1Brown video.

Example

Diagonalize the matrix if possible. $A = \begin{bmatrix} 8 & -6 \\ 9 & -7 \end{bmatrix}$.

Characteristic eqn

$\lambda^2 - \lambda - 2 = 0 \qquad \lambda_1 = 2 \qquad \lambda_2 = -1$

For $\lambda_1 = 2$, an eigenvector is $\vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$x_1 = \frac{2}{3} x_2$

For $\lambda_2 = -1$ $\qquad \vec{x} = x_2 \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$

Let $\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ by letting $x_2 = 3$.

Letting $P = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ $\det(P) = 1$

Then $P^{-1} = \dfrac{1}{1} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

$$D = P^{-1}AP = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 8 & -6 \\ 9 & -7 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

* Note: we can use any eigenvector for $\lambda_2 = -1$.
So it's correct to use $\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Our choice.

Example Continued...

Find A^6 where $A = \begin{bmatrix} 8 & -6 \\ 9 & -7 \end{bmatrix}$.

Since $D = P^{-1}AP$

$A = PDP^{-1}$ so

$A^6 = PD^6P^{-1}$

$D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, $D^6 = \begin{bmatrix} 2^6 & 0 \\ 0 & (-1)^6 \end{bmatrix} = \begin{bmatrix} 64 & 0 \\ 0 & 1 \end{bmatrix}$

$A^6 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 64 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 190 & -126 \\ 189 & -125 \end{bmatrix}$.