April 12 Math 3260 sec. 52 Spring 2024

Section 6.1: Inner Product, Length, and Orthogonality

Definition

For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ we define the **inner product** of $\mathbf{u}$ and $\mathbf{v}$ (also called the **dot product**) by the **matrix product**

$$\mathbf{u}^T\mathbf{v} = [u_1\ u_2\ \cdots\ u_n]\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Remark: Note that this product produces a scalar. It is sometimes called a *scalar product*. There are several notations for this:

$$\mathbf{u}^T\mathbf{v} = \mathbf{u}\cdot\mathbf{v} = \langle \mathbf{u}, \mathbf{v}\rangle.$$

Inner Product Properties

The dot product is an example of a class of functions that take two elements of a Real vector space and assign a scalar value. An **Inner Product** must satisfy the following properties.

Inner Product Properties

For all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and any scalar c

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ (commutitivity)
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ (distributive property)
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$ (factoring)
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$ (positive definiteness)

The Norm

That last property, being positive definite, allows us to define a **norm**.

Definition

The **norm** of the vector $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ is the nonnegative number
$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Remark: This is sometimes called the 2-norm, and might be written like $\|\mathbf{v}\|_2$. It corresponds to what we traditionally think of as *length* of a vector as a directed line segment.

Remark: This norm is often referred to as the **magnitude** of a vector.

Theorem

Theorem

For vector $\mathbf{v}$ in $\mathbb{R}^n$ and scalar c

$$\|c\mathbf{v}\| = |c|\|\mathbf{v}\|.$$

Note that $\|c\vec{v}\|^2 = (c\vec{v})\cdot(c\vec{v})$

$= c(c)\vec{v}\cdot\vec{v}$

$= c^2\|\vec{v}\|^2$

$\|c\vec{v}\| = \sqrt{c^2\|\vec{v}\|^2} = |c|\,|\,\|\vec{v}\|\,|$

$= |c|\,\|\vec{v}\|$

Unit Vectors & Normalizing

Definition

A vector $\mathbf{u}$ in $\mathbb{R}^n$ for which $\|\mathbf{u}\| = 1$ is called a **unit vector.**

Example: Show that $\mathbf{x} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$ is a unit vector.

$$\|\vec{x}\|^2 = \vec{x}\cdot\vec{x} = \left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{-2}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2$$

$$= \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = \frac{6}{6} = 1$$

$$\Rightarrow \|\vec{x}\| = \sqrt{1} = 1$$

Unit Vectors & Normalizing

Remark

Given any nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$, we can find a unit vector in the direction of $\mathbf{v}$ by dividing $\mathbf{v}$ by its norm. This is called **normalizing** the vector.

Show that if $\mathbf{v}$ is nonzero, then $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector.

$$\|\vec{u}\| = \left\| \frac{1}{\|\vec{v}\|} \vec{v} \right\| = \left| \frac{1}{\|\vec{v}\|} \right| \|\vec{v}\|$$

$$= \frac{1}{\|\vec{v}\|} \|\vec{v}\| = 1$$

Distance in $\mathbb{R}^n$

Definition:

For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the **distance between u and v** is denoted by

$$\text{dist}(\mathbf{u}, \mathbf{v}),$$

and is defined by

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

Remark: This is the same as the traditional formula for distance used in $\mathbb{R}^2$ between points (x_0, y_0) and (x_1, y_1),

$$d = \sqrt{(y_1 - y_0)^2 + (x_1 - x_0)^2}.$$

Example

Find the distance between the vectors $\mathbf{u} = (4, 0, -1, 1)$ and $\mathbf{v} = (0, 0, 2, 7)$ in $\mathbb{R}^4$.

$\text{dist}(\vec{u}, \vec{v}) = \sqrt{61}$

$\vec{u} - \vec{v} = (4, 0, -3, -6)$

$4^2 + 0^2 + (-3)^2 + (-6)^2 = 61$

Orthogonality

Definition:

Two vectors, $\mathbf{u}$ and $\mathbf{v}$, are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

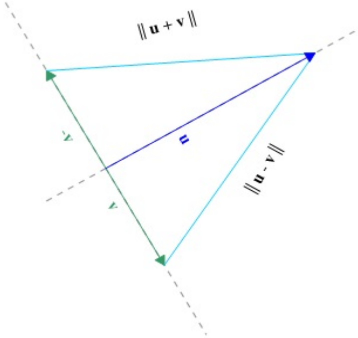

Figure: Note that two vectors are perpendicular if $\|\mathbf{u}-\mathbf{v}\| = \|\mathbf{u}+\mathbf{v}\|$

Orthogonal and Perpendicular

Show that $\|\mathbf{u}-\mathbf{v}\| = \|\mathbf{u}+\mathbf{v}\|$ if and only if $\mathbf{u}\cdot\mathbf{v} = 0$.

Consider

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

$$= \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\,\vec{u}\cdot\vec{v}.$$

$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\,\vec{u}\cdot\vec{v}.$$

If $\vec{u}\cdot\vec{v} = 0$, then $\|\vec{u}-\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2$.

making $\|\vec{u} - \vec{v}\| = \|\vec{u} + \vec{v}\|$.

And if $\|\vec{u} - \vec{v}\| = \|\vec{u} + \vec{v}\|$, then

$$\|\vec{u} - \vec{v}\|^2 - \|\vec{u} + \vec{v}\|^2 = 0$$

$$\Rightarrow \quad -4\vec{u}\cdot\vec{v} = 0 \Rightarrow \vec{u}\cdot\vec{v} = 0$$

The Pythagorean Theorem

Theorem:

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

This follows immediately from the observation that

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}\cdot\mathbf{v}.$$

The two vectors are defined as being orthogonal precisely when $\mathbf{u}\cdot\mathbf{v} = 0$.

Orthogonal Complement

Definition:

Let W be a subspace of $\mathbb{R}^n$. A vector $\mathbf{z}$ in $\mathbb{R}^n$ is said to be **orthogonal to** W if $\mathbf{z}$ is orthogonal to every vector in W. That is, if

$$\mathbf{z}\cdot\mathbf{w}=0 \quad \text{for every} \quad \mathbf{w}\in W.$$

Definition:

Given a subspace W of $\mathbb{R}^n$, the set of all vectors orthogonal to W is called the **orthogonal complement** of W and is denoted by $W^\perp$ (read as “W perp”).

$$W^\perp=\left\{\mathbf{x}\in\mathbb{R}^n \,|\, \mathbf{x}\cdot\mathbf{w}=0 \quad \text{for every} \quad \mathbf{w}\in W\right\}$$

Theorem:

Theorem:

If W is a subspace of $\mathbb{R}^n$, then $W^\perp$ is a subspace of $\mathbb{R}^n$.

This is readily proved by appealing to the properties of the inner product. In particular

$$\begin{array}{ll} (1) & \mathbf{0}\cdot\mathbf{w} = 0 \quad \text{for any vector} \quad \mathbf{w} \\ (2) & (\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w} \quad \text{and} \\ (3) & (c\mathbf{u})\cdot\mathbf{w} = c\mathbf{u}\cdot\mathbf{w}. \end{array}$$

(1) The zero vector is in $W^\perp$.

(2) If $\mathbf{u}$ and $\mathbf{v}$ are in $W^\perp$, then so is $\mathbf{u}+\mathbf{v}$.

(3) If $\mathbf{u}$ is in $W^\perp$, then so is $c\mathbf{u}$ for any scalar c.

Example

$\vec{e}_1$ $\vec{e}_3$

Let W =Span$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Then $W^{\perp}$ =Span$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

A vector in W has the form

$$\mathbf{w} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}.$$

A vector in $\mathbf{v}$ in $W^{\perp}$ has the form

$$\mathbf{v} = y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}.$$

Note that

$$\mathbf{w} \cdot \mathbf{v} = x(0) + 0(y) + z(0) = 0.$$

W is the xz-plane and $W^{\perp}$ is the y-axis in $\mathbb{R}^3$.

Example

Let $A = \begin{bmatrix} 1 & 3 & 2 \\ -2 & 0 & 4 \end{bmatrix}$. Show that if $\mathbf{x}$ is in Nul(A), then $\mathbf{x}$ is in $[\text{Row}(A)]^\perp$.

We have to show that, if $\vec{x}$ is in Nul(A), then $\vec{x}\cdot\vec{u} = 0$ for any vector $\vec{u}$ in Row(A). Let's characterize Nul A and Row (A).

$$\text{Row}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} \right\}.$$

For Nul(A), row reduce $[A\ \vec{0}]$.

$$[A\ \vec{0}] = \begin{bmatrix} 1 & 3 & 2 & 0 \\ -2 & 0 & 4 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 4/3 & 0 \end{bmatrix}$$

For $\vec{x}$ in Nul A, $\quad x_1 = 2x_3$

$x_2 = -4/3\, x_3$

x_3 free

$$\vec{x} = x_3 \begin{bmatrix} 2 \\ -4/3 \\ 1 \end{bmatrix} = \frac{x_3}{3} \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix} \qquad \text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix} \right\}.$$

Let $\vec{x}$ be in Nul A, $\quad \vec{x} = t \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix}$ and

$\vec{u}$ be in Row (A), $\quad \vec{u} = c_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$

$$\vec{u} \cdot \vec{x} = \left(c_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} \right) \cdot \left(t \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix} \right)$$

$$= c_1 t \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix} + c_2 t \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix}$$

$$= c_1 t (6 - 12 + 6) + c_2 t (-12 + 0 + 12)$$

$$= c_1 t (0) + c_2 t (0) = 0$$

so $\vec{x}$ is orthogonal to Row A.

$$A\vec{x} = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The Fundamental Subspaces of a Matrix

Theorem:

Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A. That is

$$[\mathrm{Row}(A)]^{\perp} = \mathrm{Nul}(A).$$

The orthogonal complement of the column space of A is the null space of A^T—i.e.

$$[\mathrm{Col}(A)]^{\perp} = \mathrm{Nul}(A^T).$$

Example: Find an orthogonal complement.

Let $W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 4 \\ 1 \\ -10 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ -1 \\ 13 \end{bmatrix} \right\}$. Find a basis for $W^\perp$.

We can introduce a matrix, A, such that $W = \text{Row}(A)$.

Let $A = \begin{bmatrix} 2 & 4 & 1 & -10 \\ -3 & -6 & -1 & 13 \end{bmatrix}$.

Then $W^\perp = \text{Nul}(A)$.

$$[A \ \vec{0}] = \begin{bmatrix} 2 & 4 & 1 & -10 & 0 \\ -3 & -6 & -1 & 13 & 0 \end{bmatrix} \xrightarrow{\text{rref}}$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 1 & -4 & 0 \end{bmatrix}.$$

$x_1 = -2x_2 + 3x_4$

$x_3 = 4x_4$

x_2, x_4 - free

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

A basis for $W^\perp$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$.

Note: A quick check on this work is to see whether the vectors in our basis for "W perp" are orthogonal to the vectors in the original spanning set for W. Turns out they are. If they were not, we'd have to look for an error.

If $\vec{w}_1 = \begin{bmatrix} 2 \\ 4 \\ 1 \\ -10 \end{bmatrix}$, $\vec{w}_2 = \begin{bmatrix} -3 \\ -6 \\ -1 \\ 13 \end{bmatrix}$, $\vec{z}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{z}_2 = \begin{bmatrix} 3 \\ 0 \\ 4 \\ 1 \end{bmatrix}$.

then $\vec{w}_1 \cdot \vec{z}_1 = 0$, $\vec{w}_1 \cdot \vec{z}_2 = 0$

$\vec{w}_2 \cdot \vec{z}_1 = 0$ and $\vec{w}_2 \cdot \vec{z}_2 = 0$