April 12 Math 3260 sec. 51 Spring 2024

Section 6.2: Orthogonal Sets

Remark: We know that if $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for a subspace W of $\mathbb{R}^n$, then each vector $\mathbf{x}$ in W can be realized (uniquely) as a sum

$$\mathbf{x} = c_1\mathbf{b}_2 + \cdots + c_p\mathbf{b}_p.$$

If n is very large, the computations needed to determine the coefficients $c_1, \ldots, c_p$ may require a lot of time (and machine memory).

Question: Can we seek a basis whose nature simplifies this task? And what properties should such a basis possess?

Orthogonal Sets

Definition:

An indexed set $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is said to be an **orthogonal set** provided each pair of distinct vectors in the set is orthogonal. That is, provided

$$\mathbf{u}_i \cdot \mathbf{u}_j = 0 \quad \text{whenever} \quad i \neq j.$$

Example: Show that the set $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix} \right\}$ is an orthogonal set.

Call these $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ in the given order. We'll compute

$\vec{u}_1 \cdot \vec{u}_2$, $\vec{u}_1 \cdot \vec{u}_3$ and $\vec{u}_2 \cdot \vec{u}_3$

$$\left\{ \overset{\vec{u}_1}{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}}, \overset{\vec{u}_2}{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}, \overset{\vec{u}_3}{\begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix}} \right\}$$

$$\vec{u}_1 \cdot \vec{u}_2 = 3(-1) + 1(2) + 1(1) = -3 + 2 + 1 = 0$$

$$\vec{u}_1 \cdot \vec{u}_3 = 3(-1) + 1(-4) + 1(7) = -3 - 4 + 7 = 0$$

$$\vec{u}_2 \cdot \vec{u}_3 = (-1)(-1) + 2(-4) + 1(7) = 1 - 8 + 7 = 0$$

Orthongal Basis

Definition:

An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis that is also an orthogonal set.

Theorem:

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. Then each vector $\mathbf{y}$ in W can be written as the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p, \quad \text{where the weights} \quad c_j = \frac{\mathbf{y}\cdot\mathbf{u}_j}{\mathbf{u}_j\cdot\mathbf{u}_j}.$$

Remark: What's nice about this is how simple the formula for the c's is.

Example

$$\left\{ \underset{\vec{u}_1}{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}}, \underset{\vec{u}_2}{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}, \underset{\vec{u}_3}{\begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix}} \right\}$$ is an orthogonal basis of $\mathbb{R}^3$. Express the vector $\mathbf{y} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$ as a linear combination of the basis vectors.

we need $\vec{y}\cdot\vec{u}_i$ and $\vec{u}_i\cdot\vec{u}_i$.

$\vec{y}\cdot\vec{u}_1 = -2(3)+3(1)+0 = -3$

$\vec{y}\cdot\vec{u}_2 = -2(-1)+3(2)+0 = 8$

$\vec{y}\cdot\vec{u}_3 = -2(-1)+3(-4)+0 = -10$

$\vec{u}_1\cdot\vec{u}_1 = 3^2+1^2+1^2 = 11$

$\vec{u}_2\cdot\vec{u}_2 = (-1)^2+2^2+1^2 = 6$

$\vec{u}_3\cdot\vec{u}_3 = (-1)^2+(-4)^2+7^2$

$= 66$

$$\vec{y} = \frac{-3}{11}\vec{u}_1 + \frac{8}{6}\vec{u}_2 + \frac{-10}{66}\vec{u}_3$$

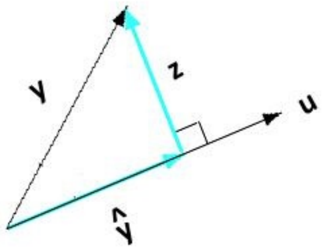
$$= \frac{-3}{11}\vec{u}_1 + \frac{4}{3}\vec{u}_2 - \frac{5}{33}\vec{u}_3$$

Projection

Given a nonzero vector $\mathbf{u}$, suppose we wish to decompose another nonzero vector $\mathbf{y}$ into a sum of the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

in such a way that $\hat{\mathbf{y}}$ is parallel to $\mathbf{u}$ and $\mathbf{z}$ is perpendicular to $\mathbf{u}$.



Projection

Since $\hat{\mathbf{y}}$ is parallel to $\mathbf{u}$, there is a scalar α such that

$$\hat{\mathbf{y}} = \alpha \mathbf{u}.$$

Find a formula for this scalar.

Assume $\vec{y} = \hat{y} + \vec{z}$ where $\hat{y} = \alpha \vec{u}$ and $\vec{z} \perp \vec{u}$

Set $\vec{y} = \hat{y} + \vec{z} = \alpha \vec{u} + \vec{z}$. Dot w/ $\vec{u}$.

$$\vec{u} \cdot \vec{y} = \vec{u} \cdot (\alpha \vec{u} + \vec{z}) = \alpha\, \vec{u} \cdot \vec{u} + \underbrace{\vec{u} \cdot \vec{z}}_{0}$$

$$\vec{u} \cdot \vec{y} = \alpha\, \vec{u} \cdot \vec{u} \quad \text{since } \vec{u} \cdot \vec{u} \neq 0$$

$$\therefore \quad \alpha = \frac{\vec{u} \cdot \vec{y}}{\vec{u} \cdot \vec{u}} = \frac{\vec{u} \cdot \vec{y}}{\|\vec{u}\|^2}.$$

$$\hat{y} = \left(\frac{\vec{u} \cdot \vec{y}}{\|\vec{u}\|^2} \right) \vec{u}$$

Projection onto the subspace $L = \text{Span}\{\mathbf{u}\}$

Projection Notation

We'll use the following notation for the project of a vector $\mathbf{y}$ onto the line $L = \text{Span}\{\mathbf{u}\}$ for nonzero vector $\mathbf{u}$.

$$\hat{\mathbf{y}} = \text{proj}_L \mathbf{y} = \left(\frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u}.$$

This may also be written as $\text{proj}_{\mathbf{u}} \mathbf{y}$.

This is read as "the projection of $\mathbf{y}$ onto $\mathbf{u}$ (or onto L)."

Example

Let $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Write $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ where $\hat{\mathbf{y}}$ is in Span$\{\mathbf{u}\}$ and $\mathbf{z}$ is orthogonal to $\mathbf{u}$.

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$$

$$\vec{y}\cdot\vec{u} = 7(4) + 6(2) = 28 + 12 = 40$$

$$\vec{u}\cdot\vec{u} = 4^2 + 2^2 = 20$$

$$\hat{y} = \frac{40}{20}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\vec{y} = \underbrace{\begin{bmatrix} 8 \\ 4 \end{bmatrix}}_{\hat{y}} + \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\vec{z}}$$

Note that $\vec{z} \cdot \vec{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = -4 + 4 = 0$

Example Continued...

Determine the distance between the point $(7, 6)$ and the line Span$\{\mathbf{u}\}$.

The distance is $\|\vec{z}\|$.

$$\text{dist}(L, \vec{y}) = \left\| \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\|$$

$$= \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

Distance between point and line

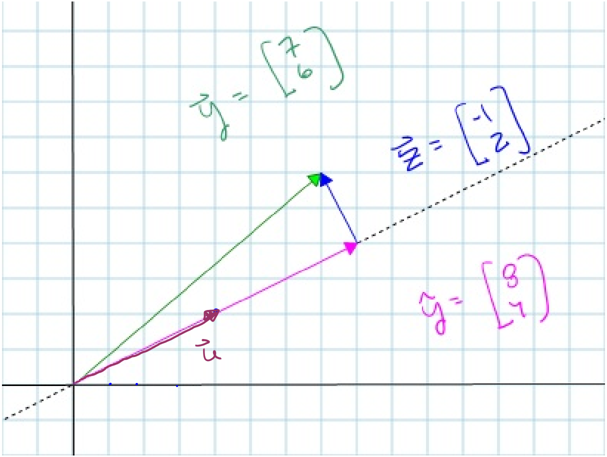


Figure: The distance between the point $(7, 6)$ and the line Span$\{\mathbf{u}\}$ is the norm of $\mathbf{z}$.

Orthonormal Sets

Definition:

A set $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is called an **orthonormal set** if it is an orthogonal set of **unit vectors.**

Definition:

An **orthonormal basis** of a subspace W of $\mathbb{R}^n$ is a basis that is also an orthonormal set.

Remark: So an **orthonormal** set (or basis) is an orthogonal set (or basis) with the extra condition that each vector has norm $\sqrt{\mathbf{u}_i \cdot \mathbf{u}_i} = 1$.

Remark: Any orthogonal set can be normalized to obtain an orthonormal one.