# April 22 Math 3260 sec. 52 Spring 2024

## Section 6.4: Gram-Schmidt Orthogonalization

We saw that the orthogonal decomposition theorem says that if $W$ is a nonzero subspace of $\mathbb{R}^n$. Each vector $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely as a sum

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}}$ is in $W$ and $\mathbf{z}$ is in $W^\perp$.

We have a formula for computing $\mathrm{proj}_W \mathbf{y}$, but it requires an orthogonal basis for $W$.

**Big Question:**

Given any-old basis for a subspace $W$ of $\mathbb{R}^n$, can we construct an orthogonal basis for that same space?

## Theorem: Gram Schmidt Process

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be any basis for the nonzero subspace $W$ of $\mathbb{R}^n$. Define the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ via

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \sum_{j=1}^{p-1}\left(\frac{\mathbf{x}_p \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}\right)\mathbf{v}_j.
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for $W$. Moreover, for each $k = 1, \ldots, p$

$$\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \mathrm{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}.$$

# Example

Find an orthonormal (that's *orthonormal* not just orthogonal) basis for Col $A$ where $A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$.

$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Let $\vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$, so

$\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is a basis for Col $A$.

Apply Gram-Schmidt.

$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$\vec{x}_2 \cdot \vec{v}_1 = -6 - 24 - 2 - 4 = -36$$

$$\vec{v}_1 \cdot \vec{v}_1 = (-1)^2 + 3^2 + 1^2 + 1^2 = 12$$

$$\vec{v}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{-36}{12} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$\vec{x}_3 \cdot \vec{v}_1 = -6+9+6-3 \quad 6$

$\vec{v}_1 \cdot \vec{v}_1 = 12$

$\vec{x}_3 \cdot \vec{v}_2 = 18+3+6+3 = 30$

$\vec{v}_2 \cdot \vec{v}_2 = 3^2+1^2+1^2+(-1)^2 = 12$

$$\vec{v}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{6}{12}\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{30}{12}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

$6 + \frac{1}{2} - \frac{15}{2} = 6-7 = -1 \qquad 6 - \frac{1}{2} - \frac{5}{2} = 6 - \frac{6}{2} = 3$

$3 - \frac{3}{2} - \frac{5}{2} = 3-4 = -1 \qquad -3 - \frac{1}{2} + \frac{5}{2} = -3+2 = -1$

New basis : $\vec{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$ , $\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ , $\vec{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$

$\vec{v}_1 \cdot \vec{v}_2 = -3+3+1-1 = 0$

$\vec{v}_1 \cdot \vec{v}_3 = 1-3+3-1 = 0$

$\vec{v}_2 \cdot \vec{v}_3 = -3-1+3+1 = 0$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for Col(A).

To get an orthonormal basis, let

$\vec{w}_i = \frac{1}{\|\vec{v}_i\|}\vec{v}_i$ , $i = 1, 2, 3$.

$\|\vec{v}_i\| = \sqrt{12}$ for $i = 1, 2, 3.$

$$\vec{w}_1 = \frac{1}{\sqrt{12}}\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{w}_2 = \frac{1}{\sqrt{12}}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

$$\vec{w}_3 = \frac{1}{\sqrt{12}}\begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}.$$

$\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is an orthonormal basis for $\mathrm{Col}(A)$.