Section 6.7 L'Hôpital's Rule

APEX CALCULUS II

Version 2.0

Authors

Gregory Hartman, Ph.D.

Department of Applied Mathematics

Virginia Military Institute

Brian Heinold, Ph.D.

Department of Mathematics and Computer Science

Mount Saint Mary's University

Troy Siemers, Ph.D.

Department of Applied Mathematics

Virginia Military Institute

Dimplekumar Chalishajar, Ph.D.

Department of Applied Mathematics

Virginia Military Institute

Editor

Jennifer Bowen, Ph.D.

Department of Mathematics and Computer Science

The College of Wooster

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6.7 L'Hôpital's Rule

While this chapter is devoted to learning techniques of integration, this section is not about integration. Rather, it is concerned with a technique of evaluating certain limits that will be useful in the following section, where integration is once more discussed.

Our treatment of limits exposed us to "0/0", an indeterminate form. If $\lim\limits_{x\to c} f(x) = 0$ and $\lim\limits_{x\to c} g(x) = 0$, we do not conclude that $\lim\limits_{x\to c} f(x)/g(x)$ is $0/0$; rather, we use $0/0$ as notation to describe the fact that both the numerator and denominator approach 0. The expression 0/0 has no numeric value; other work must be done to evaluate the limit.

Other indeterminate forms exist; they are: ∞/∞, $0\cdot\infty$, $\infty-\infty$, 0^0, 1^∞ and ∞^0. Just as "0/0" does not mean "divide 0 by 0," the expression "∞/∞" does not mean "divide infinity by infinity." Instead, it means "a quantity is growing without bound and is being divided by another quantity that is growing without bound." We cannot determine from such a statement what value, if any, results in the limit. Likewise, "$0\cdot\infty$" does not mean "multiply zero by infinity." Instead, it means "one quantity is shrinking to zero, and is being multiplied by a quantity that is growing without bound." We cannot determine from such a description what the result of such a limit will be.

This section introduces l'Hôpital's Rule, a method of resolving limits that produce the indeterminate forms 0/0 and ∞/∞. We'll also show how algebraic manipulation can be used to convert other indeterminate expressions into one of these two form so that our new rule can be applied.

Theorem 49 L'Hôpital's Rule, Part 1

Let $\lim\limits_{x\to c} f(x) = 0$ and $\lim\limits_{x\to c} g(x) = 0$, where f and g are differentiable functions on an open interval I containing c, and $g'(x) \neq 0$ on I except possibly at c. Then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

We demonstrate the use of l'Hôpital's Rule in the following examples; we will often use "LHR" as an abbreviation of "l'Hôpital's Rule."

Notes:

Example 187 **Using l'Hôpital's Rule**

Evaluate the following limits, using l'Hôpital's Rule as needed.

1. $\lim_{x\to 0} \frac{\sin x}{x}$
2. $\lim_{x\to 1} \frac{\sqrt{x+3}-2}{1-x}$
3. $\lim_{x\to 0} \frac{x^2}{1-\cos x}$
4. $\lim_{x\to 2} \frac{x^2+x-6}{x^2-3x+2}$

Solution

1. We proved this limit is 1 in Example 12 using the Squeeze Theorem. Here we use l'Hôpital's Rule to show its power.
$$\lim_{x\to 0} \frac{\sin x}{x} \overset{\text{by LHR}}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1.$$
2. $$\lim_{x\to 1} \frac{\sqrt{x+3}-2}{1-x} \overset{\text{by LHR}}{=} \lim_{x\to 1} \frac{\frac{1}{2}(x+3)^{-1/2}}{-1} = -\frac{1}{4}.$$
3. $$\lim_{x\to 0} \frac{x^2}{1-\cos x} \overset{\text{by LHR}}{=} \lim_{x\to 0} \frac{2x}{\sin x}.$$
This latter limit also evaluates to the 0/0 indeterminate form. To evaluate it, we apply l'Hôpital's Rule again.
$$\lim_{x\to 0} \frac{2x}{\sin x} \overset{\text{by LHR}}{=} \frac{2}{\cos x} = 2.$$
Thus $\lim_{x\to 0} \frac{x^2}{1-\cos x} = 2.$
4. We already know how to evaluate this limit; first factor the numerator and denominator. We then have:
$$\lim_{x\to 2} \frac{x^2+x-6}{x^2-3x+2} = \lim_{x\to 2} \frac{(x-2)(x+3)}{(x-2)(x-1)} = \lim_{x\to 2} \frac{x+3}{x-1} = 5.$$
We now show how to solve this using l'Hôpital's Rule.
$$\lim_{x\to 2} \frac{x^2+x-6}{x^2-3x+2} \overset{\text{by LHR}}{=} \lim_{x\to 2} \frac{2x+1}{2x-3} = 5.$$

The following theorem extends our initial version of l'Hôpital's Rule in two ways. It allows the technique to be applied to the indeterminate form ∞/∞ and to limits where x approaches $\pm\infty$.

Notes:

Theorem 50 L'Hôpital's Rule, Part 2

1. Let $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, where f and g are differentiable on an open interval I containing a. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

2. Let f and g be differentiable functions on the open interval (a,∞) for some value a, where $g'(x) \neq 0$ on (a,∞) and $\lim_{x\to\infty} f(x)/g(x)$ returns either 0/0 or ∞/∞. Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

A similar statement can be made for limits where x approaches $-\infty$.

Example 188 Using l'Hôpital's Rule with limits involving ∞

Evaluate the following limits.

1. $\displaystyle\lim_{x\to\infty} \frac{3x^2-100x+2}{4x^2+5x-1000}$
2. $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^3}$.

Solution

1. We can evaluate this limit already using Theorem 11; the answer is 3/4. We apply l'Hôpital's Rule to demonstrate its applicability.

$$\lim_{x\to\infty} \frac{3x^2-100x+2}{4x^2+5x-1000} \overset{\text{by LHR}}{=} \lim_{x\to\infty} \frac{6x-100}{8x+5} \overset{\text{by LHR}}{=} \lim_{x\to\infty} \frac{6}{8} = \frac{3}{4}.$$

2. $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^3} \overset{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{3x^2} \overset{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{6x} \overset{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{6} = \infty.$

   Recall that this means that the limit does not exist; as x approaches ∞, the expression e^x/x^3 grows without bound. We can infer from this that e^x grows "faster" than x^3; as x gets large, e^x is far larger than x^3. (This has important implications in computing when considering efficiency of algorithms.)

Notes:

Indeterminate Forms $0 \cdot \infty$ and $\infty - \infty$

L'Hôpital's Rule can only be applied to ratios of functions. When faced with an indeterminate form such as $0 \cdot \infty$ or $\infty - \infty$, we can sometimes apply algebra to rewrite the limit so that l'Hôpital's Rule can be applied. We demonstrate the general idea in the next example.

Example 189 Applying l'Hôpital's Rule to other indeterminate forms
Evaluate the following limits.

1. $\lim\limits_{x\to 0^+} x \cdot e^{1/x}$
2. $\lim\limits_{x\to 0^-} x \cdot e^{1/x}$
3. $\lim\limits_{x\to \infty} \ln(x+1) - \ln x$
4. $\lim\limits_{x\to \infty} x^2 - e^x$

Solution

1. As $x \to 0^+$, $x \to 0$ and $e^{1/x} \to \infty$. Thus we have the indeterminate form $0 \cdot \infty$. We rewrite the expression $x \cdot e^{1/x}$ as $\dfrac{e^{1/x}}{1/x}$; now, as $x \to 0^+$, we get the indeterminate form ∞/∞ to which l'Hôpital's Rule can be applied.

$$\lim_{x\to 0^+} x \cdot e^{1/x} = \lim_{x\to 0^+} \frac{e^{1/x}}{1/x} \overset{\text{by LHR}}{=} \lim_{x\to 0^+} \frac{(-1/x^2)e^{1/x}}{-1/x^2} = \lim_{x\to 0^+} e^{1/x} = \infty.$$

Interpretation: $e^{1/x}$ grows "faster" than x shrinks to zero, meaning their product grows without bound.

2. As $x \to 0^-$, $x \to 0$ and $e^{1/x} \to e^{-\infty} \to 0$. The the limit evaluates to $0 \cdot 0$ which is not an indeterminate form. We conclude then that

$$\lim_{x\to 0^-} x \cdot e^{1/x} = 0.$$

3. This limit initially evaluates to the indeterminate form $\infty - \infty$. By applying a logarithmic rule, we can rewrite the limit as

$$\lim_{x\to\infty} \ln(x+1) - \ln x = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right).$$

As $x \to \infty$, the argument of the ln term approaches ∞/∞, to which we can apply l'Hôpital's Rule.

$$\lim_{x\to\infty} \frac{x+1}{x} \overset{\text{by LHR}}{=} \frac{1}{1} = 1.$$

Notes:

Since $x \to \infty$ implies $\frac{x+1}{x} \to 1$, it follows that

$$x \to \infty \quad \text{implies} \quad \ln\left(\frac{x+1}{x}\right) \to \ln 1 = 0.$$

Thus

$$\lim_{x\to\infty} \ln(x+1) - \ln x = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right) = 0.$$

Interpretation: since this limit evaluates to 0, it means that for large x, there is essentially no difference between $\ln(x+1)$ and $\ln x$; their difference is essentially 0.

4. The limit $\lim\limits_{x\to\infty} x^2 - e^x$ initially returns the indeterminate form $\infty - \infty$. We can rewrite the expression by factoring out x^2; $x^2 - e^x = x^2\left(1 - \frac{e^x}{x^2}\right)$. We need to evaluate how e^x/x^2 behaves as $x \to \infty$:

$$\lim_{x\to\infty} \frac{e^x}{x^2} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{2x} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

Thus $\lim_{x\to\infty} x^2(1 - e^x/x^2)$ evaluates to $\infty \cdot (-\infty)$, which is not an indeterminate form; rather, $\infty \cdot (-\infty)$ evaluates to $-\infty$. We conclude that $\lim\limits_{x\to\infty} x^2 - e^x = -\infty$.

Interpretation: as x gets large, the difference between x^2 and e^x grows very large.

Indeterminate Forms 0^0, 1^∞ and ∞^0

When faced with an indeterminate form that involves a power, it often helps to employ the natural logarithmic function. The following Key Idea expresses the concept, which is followed by an example that demonstrates its use.

Key Idea 20 Evaluating Limits Involving Indeterminate Forms 0^0, 1^∞ and ∞^0

If $\lim\limits_{x\to c} \ln\big(f(x)\big) = L$, then $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} e^{\ln(f(x))} = e^L$.

Notes:

Example 190 Using l'Hôpital's Rule with indeterminate forms involving exponents

Evaluate the following limits.

1. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$
2. $\lim_{x\to 0^+} x^x$.

SOLUTION

1. This equivalent to a special limit given in Theorem 3; these limits have important applications within mathematics and finance. Note that the exponent approaches ∞ while the base approaches 1, leading to the indeterminate form 1^∞. Let $f(x) = (1+1/x)^x$; the problem asks to evaluate $\lim_{x\to\infty} f(x)$. Let's first evaluate $\lim_{x\to\infty} \ln\big(f(x)\big)$.

$$\begin{aligned}\lim_{x\to\infty}\ln\big(f(x)\big) &= \lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x\\ &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)\\ &= \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}\end{aligned}$$

This produces the indeterminate form 0/0, so we apply l'Hôpital's Rule.

$$\begin{aligned}&= \lim_{x\to\infty}\frac{\frac{1}{1+1/x}\cdot(-1/x^2)}{(-1/x^2)}\\ &= \lim_{x\to\infty}\frac{1}{1+1/x}\\ &= 1.\end{aligned}$$

Thus $\lim_{x\to\infty}\ln\big(f(x)\big) = 1$. We return to the original limit and apply Key Idea 20.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln(f(x))} = e^1 = e.$$

2. This limit leads to the indeterminate form 0^0. Let $f(x) = x^x$ and consider

Notes:

first $\lim_{x\to0^+} \ln\big(f(x)\big)$.

$$\begin{aligned}\lim_{x\to0^+} \ln\big(f(x)\big) &= \lim_{x\to0^+} \ln\left(x^x\right)\\ &= \lim_{x\to0^+} x\ln x\\ &= \lim_{x\to0^+} \frac{\ln x}{1/x}.\end{aligned}$$

This produces the indeterminate form $-\infty/\infty$ so we apply l'Hôpital's Rule.

$$\begin{aligned}&= \lim_{x\to0^+} \frac{1/x}{-1/x^2}\\ &= \lim_{x\to0^+} -x\\ &= 0.\end{aligned}$$

Thus $\lim_{x\to0^+} \ln\big(f(x)\big) = 0$. We return to the original limit and apply Key Idea 20.

$$\lim_{x\to0^+} x^x = \lim_{x\to0^+} f(x) = \lim_{x\to0^+} e^{\ln(f(x))} = e^0 = 1.$$

This result is supported by the graph of $f(x) = x^x$ given in Figure 6.17.

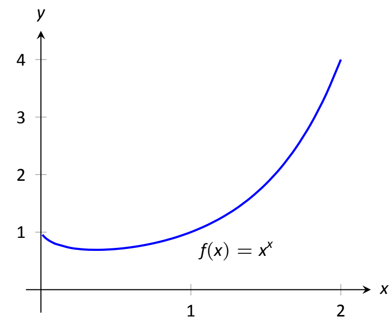

Figure 6.17: A graph of $f(x) = x^x$ supporting the fact that as $x \to 0^+$, $f(x) \to 1$.

Our brief revisit of limits will be rewarded in the next section where we consider *improper integration.* So far, we have only considered definite integrals where the bounds are finite numbers, such as $\int_0^1 f(x)\,dx$. Improper integration considers integrals where one, or both, of the bounds are "infinity." Such integrals have many uses and applications, in addition to generating ideas that are enlightening.

Notes:

Exercises 6.7

Terms and Concepts

1. List the different indeterminate forms described in this section.
2. T/F: l'Hôpital's Rule provides a faster method of computing derivatives.
3. T/F: l'Hôpital's Rule states that $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)}{g'(x)}$.
4. Explain what the indeterminate form "1^∞" means.
5. Fill in the blanks: The Quotient Rule is applied to $\frac{f(x)}{g(x)}$ when taking ________; l'Hôpital's Rule is applied when taking certain ________.
6. Create (but do not evaluate!) a limit that returns "∞^0".
7. Create a function $f(x)$ such that $\lim\limits_{x\to 1} f(x)$ returns "0^0".

Problems

In Exercises 8 – 52, evaluate the given limit.

8. $\lim\limits_{x\to 1} \frac{x^2+x-2}{x-1}$
9. $\lim\limits_{x\to 2} \frac{x^2+x-6}{x^2-7x+10}$
10. $\lim\limits_{x\to \pi} \frac{\sin x}{x-\pi}$
11. $\lim\limits_{x\to \pi/4} \frac{\sin x-\cos x}{\cos(2x)}$
12. $\lim\limits_{x\to 0} \frac{\sin(5x)}{x}$
13. $\lim\limits_{x\to 0} \frac{\sin(2x)}{x+2}$
14. $\lim\limits_{x\to 0} \frac{\sin(2x)}{\sin(3x)}$
15. $\lim\limits_{x\to 0} \frac{\sin(ax)}{\sin(bx)}$
16. $\lim\limits_{x\to 0^+} \frac{e^x-1}{x^2}$
17. $\lim\limits_{x\to 0^+} \frac{e^x-x-1}{x^2}$
18. $\lim\limits_{x\to 0^+} \frac{x-\sin x}{x^3-x^2}$
19. $\lim\limits_{x\to \infty} \frac{x^4}{e^x}$
20. $\lim\limits_{x\to \infty} \frac{\sqrt{x}}{e^x}$
21. $\lim\limits_{x\to \infty} \frac{e^x}{\sqrt{x}}$
22. $\lim\limits_{x\to \infty} \frac{e^x}{2^x}$
23. $\lim\limits_{x\to \infty} \frac{e^x}{3^x}$
24. $\lim\limits_{x\to 3} \frac{x^3-5x^2+3x+9}{x^3-7x^2+15x-9}$
25. $\lim\limits_{x\to -2} \frac{x^3+4x^2+4x}{x^3+7x^2+16x+12}$
26. $\lim\limits_{x\to \infty} \frac{\ln x}{x}$
27. $\lim\limits_{x\to \infty} \frac{\ln(x^2)}{x}$
28. $\lim\limits_{x\to \infty} \frac{\left(\ln x\right)^2}{x}$
29. $\lim\limits_{x\to 0^+} x\cdot\ln x$
30. $\lim\limits_{x\to 0^+} \sqrt{x}\cdot\ln x$
31. $\lim\limits_{x\to 0^+} xe^{1/x}$
32. $\lim\limits_{x\to \infty} x^3-x^2$
33. $\lim\limits_{x\to \infty} \sqrt{x}-\ln x$
34. $\lim\limits_{x\to -\infty} xe^x$
35. $\lim\limits_{x\to 0^+} \frac{1}{x^2}e^{-1/x}$
36. $\lim\limits_{x\to 0^+} (1+x)^{1/x}$
37. $\lim\limits_{x\to 0^+} (2x)^x$
38. $\lim\limits_{x\to 0^+} (2/x)^x$
39. $\lim\limits_{x\to 0^+} (\sin x)^x$ Hint: use the Squeeze Theorem.
40. $\lim\limits_{x\to 1^+} (1-x)^{1-x}$
41. $\lim\limits_{x\to \infty} (x)^{1/x}$
42. $\lim\limits_{x\to \infty} (1/x)^x$
43. $\lim\limits_{x\to 1^+} (\ln x)^{1-x}$
44. $\lim\limits_{x\to \infty} (1+x)^{1/x}$
45. $\lim\limits_{x\to \infty} (1+x^2)^{1/x}$
46. $\lim\limits_{x\to \pi/2} \tan x\cos x$
47. $\lim\limits_{x\to \pi/2} \tan x\sin(2x)$
48. $\lim\limits_{x\to 1^+} \frac{1}{\ln x}-\frac{1}{x-1}$
49. $\lim\limits_{x\to 3^+} \frac{5}{x^2-9}-\frac{x}{x-3}$
50. $\lim\limits_{x\to \infty} x\tan(1/x)$
51. $\lim\limits_{x\to \infty} \frac{(\ln x)^3}{x}$
52. $\lim\limits_{x\to 1} \frac{x^2+x-2}{\ln x}$

Solutions to Odd Exercises

21. $-\frac{1}{\sqrt{x^2+9}}+C$ (Trig. Subst. is not needed)
23. $\frac{1}{18}\frac{x+2}{x^2+4x+13}+\frac{1}{54}\tan^{-1}\left(\frac{x+2}{2}\right)+C$
25. $\frac{1}{7}\left(-\frac{\sqrt{5-x^2}}{x}-\sin^{-1}(x/\sqrt{5})\right)+C$
27. $\pi/2$
29. $2\sqrt{2}+2\ln(1+\sqrt{2})$
31. $9\sin^{-1}(1/3)+\sqrt{8}$ Note: the new lower bound is $\theta=\sin^{-1}(-1/3)$ and the new upper bound is $\theta=\sin^{-1}(1/3)$. The final answer comes with recognizing that $\sin^{-1}(-1/3)=-\sin^{-1}(1/3)$ and that $\cos\left(\sin^{-1}(1/3)\right)=\cos\left(\sin^{-1}(-1/3)\right)=\sqrt{8}/3$.

Section 6.5

1. rational
3. $\frac{A}{x}+\frac{B}{x-3}$
5. $\frac{A}{x-\sqrt{7}}+\frac{B}{x+\sqrt{7}}$
7. $3\ln|x-2|+4\ln|x+5|+C$
9. $\frac{1}{3}(\ln|x+2|-\ln|x-2|)+C$
11. $-\frac{4}{x+8}-3\ln|x+8|+C$
13. $-\ln|2x-3|+5\ln|x-1|+2\ln|x+3|+C$
15. $x+\ln|x-1|-\ln|x+2|+C$
17. $2x+C$
19. $-\frac{3}{2}\ln\left|x^2+4x+10\right|+x+\frac{\tan^{-1}\left(\frac{x+2}{\sqrt{6}}\right)}{\sqrt{6}}+C$
21. $2\ln|x-3|+2\ln|x^2+6x+10|-4\tan^{-1}(x+3)+C$
23. $\frac{1}{2}\left(3\ln\left|x^2+2x+17\right|-4\ln|x-7|+\tan^{-1}\left(\frac{x+1}{4}\right)\right)+C$
25. $\frac{1}{2}\ln\left|x^2+10x+27\right|+5\ln|x+2|-6\sqrt{2}\tan^{-1}\left(\frac{x+5}{\sqrt{2}}\right)+C$
27. $5\ln(9/4)-\frac{1}{3}\ln(17/2)\approx 3.3413$
29. $1/8$

Section 6.6

1. Because $\cosh x$ is always positive.
3. $$\begin{aligned}\coth^2 x-\operatorname{csch}^2 x &= \left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)^2-\left(\frac{2}{e^x-e^{-x}}\right)^2\\ &= \frac{(e^{2x}+2+e^{-2x})-(4)}{e^{2x}-2+e^{-2x}}\\ &= \frac{e^{2x}-2+e^{-2x}}{e^{2x}-2+e^{-2x}}\\ &= 1\end{aligned}$$
5. $$\begin{aligned}\cosh^2 x &= \left(\frac{e^x+e^{-x}}{2}\right)^2\\ &= \frac{e^{2x}+2+e^{-2x}}{4}\\ &= \frac{1}{2}\frac{(e^{2x}+e^{-2x})+2}{2}\\ &= \frac{1}{2}\left(\frac{e^{2x}+e^{-2x}}{2}+1\right)\\ &= \frac{\cosh 2x+1}{2}.\end{aligned}$$
7. $$\begin{aligned}\frac{d}{dx}[\operatorname{sech} x] &= \frac{d}{dx}\left[\frac{2}{e^x+e^{-x}}\right]\\ &= \frac{-2(e^x-e^{-x})}{(e^x+e^{-x})^2}\\ &= -\frac{2(e^x-e^{-x})}{(e^x+e^{-x})(e^x+e^{-x})}\\ &= -\frac{2}{e^x+e^{-x}}\cdot\frac{e^x-e^{-x}}{e^x+e^{-x}}\\ &= -\operatorname{sech} x\tanh x\end{aligned}$$
9. $$\int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}\,dx$$
Let $u=\cosh x$; $du=(\sinh x)dx$
$$\begin{aligned}&= \int \frac{1}{u}\,du\\ &= \ln|u|+C\\ &= \ln(\cosh x)+C.\end{aligned}$$
11. $2\sinh 2x$
13. $\coth x$
15. $x\cosh x$
17. $\frac{3}{\sqrt{9x^2+1}}$
19. $\frac{1}{1-(x+5)^2}$
21. $\sec x$
23. $y=3/4(x-\ln 2)+5/4$
25. $y=x$
27. $1/2\ln(\cosh(2x))+C$
29. $1/2\sinh^2 x+C$ or $1/2\cosh^2 x+C$
31. $x\cosh(x)-\sinh(x)+C$
33. $\cosh^{-1}(x^2/2)+C=\ln(x^2+\sqrt{x^4-4})+C$
35. $\frac{1}{16}\tan^{-1}(x/2)+\frac{1}{32}\ln|x-2|+\frac{1}{32}\ln|x+2|+C$
37. $\tan^{-1}(e^x)+C$
39. $x\tanh^{-1}x+1/2\ln|x^2-1|+C$
41. 0
43. 2

Section 6.7

1. $0/0, \infty/\infty, 0\cdot\infty, \infty-\infty, 0^0, 1^\infty, \infty^0$
3. F
5. derivatives; limits
7. Answers will vary.
9. $-5/3$
11. $-\sqrt{2}/2$
13. 0
15. a/b
17. $1/2$
19. 0
21. ∞
23. 0
25. -2
27. 0
29. 0

31. ∞

33. ∞

35. 0

37. 1

39. 1

41. 1

43. 1

45. 1

47. 2

49. $-\infty$

51. 0

Section 6.8

1. The interval of integration is finite, and the integrand is continuous on that interval.

3. converges; could also state < 10.

5. $p > 1$

7. $e^5/2$

9. $1/3$

11. $1/\ln 2$

13. diverges

15. 1

17. diverges

19. diverges

21. diverges

23. 1

25. 0

27. $-1/4$

29. -1

31. diverges

33. $1/2$

35. converges; Limit Comparison Test with $1/x^{3/2}$.

37. converges; Direct Comparison Test with xe^{-x}.

39. converges; Direct Comparison Test with xe^{-x}.

41. diverges; Direct Comparison Test with $x/(x^2 + \cos x)$.

43. converges; Limit Comparison Test with $1/e^x$.

Chapter 7

Section 7.1

1. T

3. Answers will vary.

5. $16/3$

7. π

9. $2\sqrt{2}$

11. 4.5

13. $2 - \pi/2$

15. $1/6$

17. On regions such as $[\pi/6, 5\pi/6]$, the area is $3\sqrt{3}/2$. On regions such as $[-\pi/2, \pi/6]$, the area is $3\sqrt{3}/4$.

19. $5/3$

21. $9/4$

23. 1

25. 4

27. 219,000 ft^2

Section 7.2

1. T

3. Recall that "dx" does not just "sit there;" it is multiplied by $A(x)$ and represents the thickness of a small slice of the solid. Therefore dx has units of in, giving $A(x)\ dx$ the units of in^3.

5. $175\pi/3$ units^3

7. $\pi/6$ units^3

9. $35\pi/3$ units^3

11. $2\pi/15$ units^3

13. (a) $512\pi/15$
    (b) $256\pi/5$
    (c) $832\pi/15$
    (d) $128\pi/3$

15. (a) $104\pi/15$
    (b) $64\pi/15$
    (c) $32\pi/5$

17. (a) 8π
    (b) 8π
    (c) $16\pi/3$
    (d) $8\pi/3$

19. The cross–sections of this cone are the same as the cone in Exercise 18. Thus they have the same volume of $250\pi/3$ units^3.

21. Orient the solid so that the x-axis is parallel to long side of the base. All cross–sections are trapezoids (at the far left, the trapezoid is a square; at the far right, the trapezoid has a top length of 0, making it a triangle). The area of the trapezoid at x is $A(x) = 1/2(-1/2x + 5 + 5)(5) = -5/4x + 25$. The volume is 187.5 units^3.

Section 7.3

1. T

3. F

5. $9\pi/2$ units^3

7. $\pi^2 - 2\pi$ units^3

9. $48\pi\sqrt{3}/5$ units^3

11. $\pi^2/4$ units^3

13. (a) $4\pi/5$
    (b) $8\pi/15$
    (c) $\pi/2$
    (d) $5\pi/6$

15. (a) $4\pi/3$
    (b) $\pi/3$
    (c) $4\pi/3$