APEX Calculus I

Version 2.0

Authors

Gregory Hartman, Ph.D.

Department of Applied Mathematics

Virginia Military Institute

Brian Heinold, Ph.D.

Department of Mathematics and Computer Science

Mount Saint Mary's University

Troy Siemers, Ph.D.

Department of Applied Mathematics

Virginia Military Institute

Dimplekumar Chalishajar, Ph.D.

Department of Applied Mathematics

Virginia Military Institute

Editor

Jennifer Bowen, Ph.D.

Department of Mathematics and Computer Science

The College of Wooster

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2.7 Derivatives of Inverse Functions

Recall that a function $y = f(x)$ is said to be *one to one* if it passes the horizontal line test; that is, for two different x values x_1 and x_2, we do *not* have $f(x_1) = f(x_2)$. In some cases the domain of f must be restricted so that it is one to one. For instance, consider $f(x) = x^2$. Clearly, $f(-1) = f(1)$, so f is not one to one on its regular domain, but by restricting f to $(0, \infty)$, f is one to one.

Now recall that one to one functions have *inverses*. That is, if f is one to one, it has an inverse function, denoted by f^{-1}, such that if $f(a) = b$, then $f^{-1}(b) = a$. The domain of f^{-1} is the range of f, and vice-versa. For ease of notation, we set $g = f^{-1}$ and treat g as a function of x.

Since $f(a) = b$ implies $g(b) = a$, when we compose f and g we get a nice result:

$$f\big(g(b)\big) = f(a) = b.$$

In general, $f\big(g(x)\big) = x$ and $g\big(f(x)\big) = x$. This gives us a convenient way to check if two functions are inverses of each other: compose them and if the result is x, then they are inverses (on the appropriate domains.)

When the point (a, b) lies on the graph of f, the point (b, a) lies on the graph of g. This leads us to discover that the graph of g is the reflection of f across the line $y = x$. In Figure 2.30 we see a function graphed along with its inverse. See how the point $(1, 1.5)$ lies on one graph, whereas $(1.5, 1)$ lies on the other. Because of this relationship, whatever we know about f can quickly be transferred into knowledge about g.

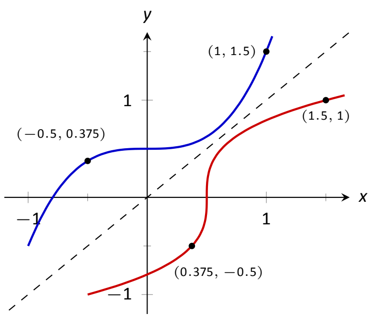

Figure 2.30: A function f along with its inverse f^{-1}. (Note how it does not matter which function we refer to as f; the other is f^{-1}.)

For example, consider Figure 2.31 where the tangent line to f at the point (a, b) is drawn. That line has slope $f'(a)$. Through reflection across $y = x$, we can see that the tangent line to g at the point (b, a) should have slope $\dfrac{1}{f'(a)}$. This then tells us that $g'(b) = \dfrac{1}{f'(a)}$.

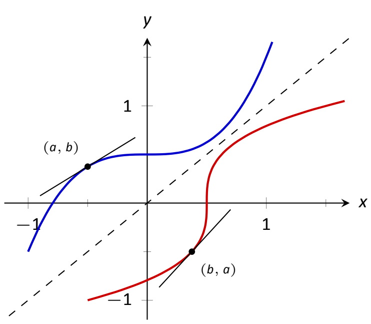

Figure 2.31: Corresponding tangent lines drawn to f and f^{-1}.

Consider:

Information about f	Information about $g = f^{-1}$
$(-0.5, 0.375)$ lies on f	$(0.375, -0.5)$ lies on g
Slope of tangent line to f at $x = -0.5$ is $3/4$	Slope of tangent line to g at $x = 0.375$ is $4/3$
$f'(-0.5) = 3/4$	$g'(0.375) = 4/3$

We have discovered a relationship between f' and g' in a mostly graphical way. We can realize this relationship analytically as well. Let $y = g(x)$, where again $g = f^{-1}$. We want to find y'. Since $y = g(x)$, we know that $f(y) = x$. Using the Chain Rule and Implicit Differentiation, take the derivative of both sides of

Notes:

this last equality.

$$\begin{aligned}\frac{d}{dx}\Big(f(y)\Big) &= \frac{d}{dx}\Big(x\Big)\\ f'(y)\cdot y' &= 1\\ y' &= \frac{1}{f'(y)}\\ y' &= \frac{1}{f'(g(x))}\end{aligned}$$

This leads us to the following theorem.

Theorem 22 Derivatives of Inverse Functions

Let f be differentiable and one to one on an open interval I, where $f'(x) \neq 0$ for all x in I, let J be the range of f on I, let g be the inverse function of f, and let $f(a) = b$ for some a in I. Then g is a differentiable function on J, and in particular,

1. $\left(f^{-1}\right)'(b) = g'(b) = \dfrac{1}{f'(a)}$ and 2. $\left(f^{-1}\right)'(x) = g'(x) = \dfrac{1}{f'(g(x))}$

The results of Theorem 22 are not trivial; the notation may seem confusing at first. Careful consideration, along with examples, should earn understanding.

In the next example we apply Theorem 22 to the arcsine function.

Example 73 Finding the derivative of an inverse trigonometric function

Let $y = \arcsin x = \sin^{-1} x$. Find y' using Theorem 22.

Solution Adopting our previously defined notation, let $g(x) = \arcsin x$ and $f(x) = \sin x$. Thus $f'(x) = \cos x$. Applying the theorem, we have

$$\begin{aligned} g'(x) &= \frac{1}{f'(g(x))}\\ &= \frac{1}{\cos(\arcsin x)}.\end{aligned}$$

This last expression is not immediately illuminating. Drawing a figure will help, as shown in Figure 2.33. Recall that the sine function can be viewed as taking in an angle and returning a ratio of sides of a right triangle, specifically, the ratio "opposite over hypotenuse." This means that the arcsine function takes as input a ratio of sides and returns an angle. The equation $y = \arcsin x$ can be rewritten as $y = \arcsin(x/1)$; that is, consider a right triangle where the

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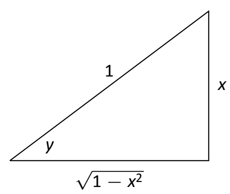


Figure 2.33: A right triangle defined by $y = \sin^{-1}(x/1)$ with the length of the third leg found using the Pythagorean Theorem.

hypotenuse has length 1 and the side opposite of the angle with measure y has length x. This means the final side has length $\sqrt{1-x^2}$, using the Pythagorean Theorem.

Therefore $\cos(\sin^{-1} x) = \cos y = \sqrt{1-x^2}/1 = \sqrt{1-x^2}$, resulting in

$$\frac{d}{dx}\big(\arcsin x\big) = g'(x) = \frac{1}{\sqrt{1-x^2}}.$$

Remember that the input x of the arcsine function is a ratio of a side of a right triangle to its hypotenuse; the absolute value of this ratio will never be greater than 1. Therefore the inside of the square root will never be negative.

In order to make $y = \sin x$ one to one, we restrict its domain to $[-\pi/2, \pi/2]$; on this domain, the range is $[-1, 1]$. Therefore the domain of $y = \arcsin x$ is $[-1, 1]$ and the range is $[-\pi/2, \pi/2]$. When $x = \pm 1$, note how the derivative of the arcsine function is undefined; this corresponds to the fact that as $x \to \pm 1$, the tangent lines to arcsine approach vertical lines with undefined slopes.

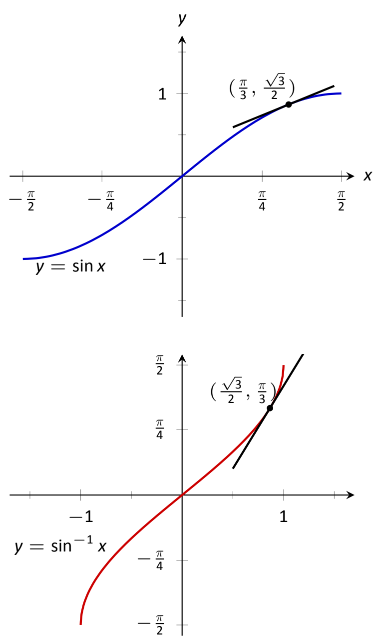


Figure 2.34: Graphs of $\sin x$ and $\sin^{-1} x$ along with corresponding tangent lines.

In Figure 2.34 we see $f(x) = \sin x$ and $f^{-1} = \sin^{-1} x$ graphed on their respective domains. The line tangent to $\sin x$ at the point $(\pi/3, \sqrt{3}/2)$ has slope $\cos \pi/3 = 1/2$. The slope of the corresponding point on $\sin^{-1} x$, the point $(\sqrt{3}/2, \pi/3)$, is

$$\frac{1}{\sqrt{1-(\sqrt{3}/2)^2}} = \frac{1}{\sqrt{1-3/4}} = \frac{1}{\sqrt{1/4}} = \frac{1}{1/2} = 2,$$

verifying yet again that at corresponding points, a function and its inverse have reciprocal slopes.

Using similar techniques, we can find the derivatives of all the inverse trigonometric functions. In Figure 2.32 we show the restrictions of the domains of the standard trigonometric functions that allow them to be invertible.

Notes:

Function	Domain	Range	Inverse Function	Domain	Range
$\sin x$	$[-\pi/2, \pi/2]$	$[-1, 1]$	$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos x$	$[0, \pi]$	$[-1, 1]$	$\cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$
$\tan x$	$(-\pi/2, \pi/2)$	$(-\infty, \infty)$	$\tan^{-1}(x)$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
$\csc x$	$[-\pi/2, 0) \cup (0, \pi/2]$	$(-\infty, -1] \cup [1, \infty)$	$\csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$
$\sec x$	$[0, \pi/2) \cup (\pi/2, \pi]$	$(-\infty, -1] \cup [1, \infty)$	$\sec^{-1}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
$\cot x$	$(0, \pi)$	$(-\infty, \infty)$	$\cot^{-1}(x)$	$(-\infty, \infty)$	$(0, \pi)$

Figure 2.32: Domains and ranges of the trigonometric and inverse trigonometric functions.

Theorem 23 Derivatives of Inverse Trigonometric Functions

The inverse trigonometric functions are differentiable on their domains (as listed in Figure 2.32) and their derivatives are as follows:

1. $\dfrac{d}{dx}\big(\sin^{-1}(x)\big) = \dfrac{1}{\sqrt{1-x^2}}$
2. $\dfrac{d}{dx}\big(\sec^{-1}(x)\big) = \dfrac{1}{|x|\sqrt{x^2-1}}$
3. $\dfrac{d}{dx}\big(\tan^{-1}(x)\big) = \dfrac{1}{1+x^2}$
4. $\dfrac{d}{dx}\big(\cos^{-1}(x)\big) = -\dfrac{1}{\sqrt{1-x^2}}$
5. $\dfrac{d}{dx}\big(\csc^{-1}(x)\big) = -\dfrac{1}{|x|\sqrt{x^2-1}}$
6. $\dfrac{d}{dx}\big(\cot^{-1}(x)\big) = -\dfrac{1}{1+x^2}$

Note how the last three derivatives are merely the opposites of the first three, respectively. Because of this, the first three are used almost exclusively throughout this text.

In Section 2.3, we stated without proof or explanation that $\dfrac{d}{dx}\big(\ln x\big) = \dfrac{1}{x}$. We can justify that now using Theorem 22, as shown in the example.

Example 74 Finding the derivative of $y = \ln x$

Use Theorem 22 to compute $\dfrac{d}{dx}\big(\ln x\big)$.

Solution View $y = \ln x$ as the inverse of $y = e^x$. Therefore, using our standard notation, let $f(x) = e^x$ and $g(x) = \ln x$. We wish to find $g'(x)$. Theorem

Notes:

22 gives:

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

In this chapter we have defined the derivative, given rules to facilitate its computation, and given the derivatives of a number of standard functions. We restate the most important of these in the following theorem, intended to be a reference for further work.

Theorem 24 Glossary of Derivatives of Elementary Functions

Let u and v be differentiable functions, and let a, c and n be a real numbers, $a > 0$, $n \neq 0$.

1. $\frac{d}{dx}\big(cu\big) = cu'$
2. $\frac{d}{dx}\big(u \pm v\big) = u' \pm v'$
3. $\frac{d}{dx}\big(u \cdot v\big) = uv' + u'v$
4. $\frac{d}{dx}\big(\frac{u}{v}\big) = \frac{u'v - uv'}{v^2}$
5. $\frac{d}{dx}\big(u(v)\big) = u'(v)v'$
6. $\frac{d}{dx}\big(c\big) = 0$
7. $\frac{d}{dx}\big(x\big) = 1$
8. $\frac{d}{dx}\big(x^n\big) = nx^{n-1}$
9. $\frac{d}{dx}\big(e^x\big) = e^x$
10. $\frac{d}{dx}\big(a^x\big) = \ln a \cdot a^x$
11. $\frac{d}{dx}\big(\ln x\big) = \frac{1}{x}$
12. $\frac{d}{dx}\big(\log_a x\big) = \frac{1}{\ln a} \cdot \frac{1}{x}$
13. $\frac{d}{dx}\big(\sin x\big) = \cos x$
14. $\frac{d}{dx}\big(\cos x\big) = -\sin x$
15. $\frac{d}{dx}\big(\csc x\big) = -\csc x \cot x$
16. $\frac{d}{dx}\big(\sec x\big) = \sec x \tan x$
17. $\frac{d}{dx}\big(\tan x\big) = \sec^2 x$
18. $\frac{d}{dx}\big(\cot x\big) = -\csc^2 x$
19. $\frac{d}{dx}\big(\sin^{-1} x\big) = \frac{1}{\sqrt{1-x^2}}$
20. $\frac{d}{dx}\big(\cos^{-1} x\big) = -\frac{1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}\big(\csc^{-1} x\big) = -\frac{1}{|x|\sqrt{x^2-1}}$
22. $\frac{d}{dx}\big(\sec^{-1} x\big) = \frac{1}{|x|\sqrt{x^2-1}}$
23. $\frac{d}{dx}\big(\tan^{-1} x\big) = \frac{1}{1+x^2}$
24. $\frac{d}{dx}\big(\cot^{-1} x\big) = -\frac{1}{1+x^2}$

Notes:

Exercises 2.7

Terms and Concepts

1. T/F: Every function has an inverse.

2. In your own words explain what it means for a function to be "one to one."

3. If $(1, 10)$ lies on the graph of $y = f(x)$, what can be said about the graph of $y = f^{-1}(x)$?

4. If $(1, 10)$ lies on the graph of $y = f(x)$ and $f'(1) = 5$, what can be said about $y = f^{-1}(x)$?

Problems

In Exercises 5 – 8, verify that the given functions are inverses.

5. $f(x) = 2x + 6$ and $g(x) = \frac{1}{2}x - 3$

6. $f(x) = x^2 + 6x + 11$, $x \geq 3$ and
   $g(x) = \sqrt{x-2} - 3$, $x \geq 2$

7. $f(x) = \dfrac{3}{x-5}$, $x \neq 5$ and
   $g(x) = \dfrac{3+5x}{x}$, $x \neq 0$

8. $f(x) = \dfrac{x+1}{x-1}$, $x \neq 1$ and $g(x) = f(x)$

In Exercises 9 – 14, an invertible function $f(x)$ is given along with a point that lies on its graph. Using Theorem 22, evaluate $\left(f^{-1}\right)'(x)$ at the indicated value.

9. $f(x) = 5x + 10$
   Point$= (2, 20)$
   Evaluate $\left(f^{-1}\right)'(20)$

10. $f(x) = x^2 - 2x + 4$, $x \geq 1$
    Point$= (3, 7)$
    Evaluate $\left(f^{-1}\right)'(7)$

11. $f(x) = \sin 2x$, $-\pi/4 \leq x \leq \pi/4$
    Point$= (\pi/6, \sqrt{3}/2)$
    Evaluate $\left(f^{-1}\right)'(\sqrt{3}/2)$

12. $f(x) = x^3 - 6x^2 + 15x - 2$
    Point$= (1, 8)$
    Evaluate $\left(f^{-1}\right)'(8)$

13. $f(x) = \dfrac{1}{1+x^2}$, $x \geq 0$
    Point$= (1, 1/2)$
    Evaluate $\left(f^{-1}\right)'(1/2)$

14. $f(x) = 6e^{3x}$
    Point$= (0, 6)$
    Evaluate $\left(f^{-1}\right)'(6)$

In Exercises 15 – 24, compute the derivative of the given function.

15. $h(t) = \sin^{-1}(2t)$

16. $f(t) = \sec^{-1}(2t)$

17. $g(x) = \tan^{-1}(2x)$

18. $f(x) = x \sin^{-1} x$

19. $g(t) = \sin t \cos^{-1} t$

20. $f(t) = \ln te^t$

21. $h(x) = \dfrac{\sin^{-1} x}{\cos^{-1} x}$

22. $g(x) = \tan^{-1}(\sqrt{x})$

23. $f(x) = \sec^{-1}(1/x)$

24. $f(x) = \sin(\sin^{-1} x)$

In Exercises 25 – 27, compute the derivative of the given function in two ways:

(a) By simplifying first, then taking the derivative, and

(b) by using the Chain Rule first then simplifying.

Verify that the two answers are the same.

25. $f(x) = \sin(\sin^{-1} x)$

26. $f(x) = \tan^{-1}(\tan x)$

27. $f(x) = \sin(\cos^{-1} x)$

In Exercises 28 – 29, find the equation of the line tangent to the graph of f at the indicated x value.

28. $f(x) = \sin^{-1} x$ at $x = \frac{\sqrt{2}}{2}$

29. $f(x) = \cos^{-1}(2x)$ at $x = \frac{\sqrt{3}}{4}$

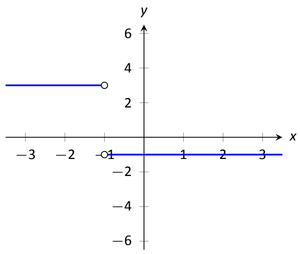
Review

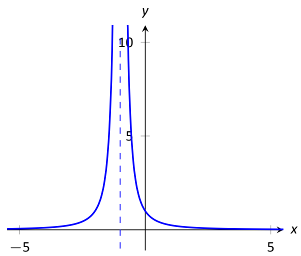
30. Find $\frac{dy}{dx}$, where $x^2y - y^2x = 1$.

31. Find the equation of the line tangent to the graph of $x^2 + y^2 + xy = 7$ at the point $(1, 2)$.

32. Let $f(x) = x^3 + x$.
    Evaluate $\displaystyle\lim_{s\to 0} \frac{f(x+s) - f(x)}{s}$.

Solutions to Odd Exercises

43. 

45. 

Section 2.5

1. T
3. F
5. T
7. $f'(t) = 15(3t-2)^4$
9. $h'(t) = (6t+1)e^{3t^2+t-1}$
11. $f'(x) = -3\sin(3x)$
13. $h'(t) = 8\sin^3(2t)\cos(2t)$
15. $f'(x) = -\tan x$
17. $f'(x) = 2/x$
19. $g'(t) = -\ln 5 \cdot 5^{\cos t}\sin t$
21. $m'(w) = \ln(3/2)(3/2)^w$
23. $f'(x) = \frac{2^{x^2}(\ln 3 \cdot 3^x x^2 2x+1)-(3^{x^2}+x)(\ln 2 \cdot 2^{x^2}2x)}{2^{2x^2}}$
25. $g'(t) = 5\cos(t^2+3t)\cos(5t-7)-(2t+3)\sin(t^2+3t)\sin(5t-7)$
27. Tangent line: $y = 0$
    Normal line: $x = 0$
29. Tangent line: $y = -3(\quad - \quad /2) + 1$
    Normal line: $y = 1/3(\quad - \quad /2) + 1$
31. In both cases the derivative is the same: $1/x$.
33. (a) ° F/mph
    (b) The sign would be negative; when the wind is blowing at 10 mph, any increase in wind speed will make it feel colder, i.e., a lower number on the Fahrenheit scale.

Section 2.6

1. Answers will vary.
3. T
5. $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$
7. $g'(t) = \sqrt{t}\cos t + \frac{\sin t}{2\sqrt{t}}$
9. $\frac{dy}{dx} = \frac{-4x^3}{2y+1}$
11. $\frac{dy}{dx} = \sin(x)\sec(y)$
13. $\frac{dy}{dx} = \frac{y}{x}$
15. $-\frac{2\sin(y)\cos(y)}{x}$
17. $\frac{1}{2y+2}$
19. $\frac{-\cos(x)(x+\cos(y))+\sin(x)+y}{\sin(y)(\sin(x)+y)+x+\cos(y)}$
21. $-\frac{2x+y}{2y+x}$
23. (a) $y = 0$
    (b) $y = -1.859(x-0.1)+0.281$
25. (a) $y = 4$
    (b) $y = 0.93(x-2)+\sqrt[4]{108}$
27. (a) $y = -\frac{1}{\sqrt{3}}(x-\frac{7}{2})+\frac{6+3\sqrt{3}}{2}$
    (b) $y = \sqrt{3}(x-\frac{4+3\sqrt{3}}{2})+\frac{3}{2}$
29. $\frac{d^2y}{dx^2} = \frac{3}{5}\frac{y^{3/5}}{x^{8/5}}+\frac{3}{5}\frac{1}{yx^{6/5}}$
31. $\frac{d^2y}{dx^2} = 0$
33. $y' = (2x)^{x^2}\left(2x\ln(2x)+x\right)$
    Tangent line: $y = (2+4\ln 2)(x-1)+2$
35. $y' = x^{\sin(x)+2}\left(\cos x\ln x+\frac{\sin x+2}{x}\right)$
    Tangent line: $y = (3 \quad ^2/4)(x- \quad /2)+(\quad /2)^3$
37. $y' = \frac{(x+1)(x+2)}{(x+3)(x+4)}\left(\frac{1}{x+1}+\frac{1}{x+2}-\frac{1}{x+3}-\frac{1}{x+4}\right)$
    Tangent line: $y = 11/72x+1/6$

Section 2.7

1. F
3. The point $(10, 1)$ lies on the graph of $y = f^{-1}(x)$ (assuming f is invertible).
5. Compose $f(g(x))$ and $g(f(x))$ to confirm that each equals x.
7. Compose $f(g(x))$ and $g(f(x))$ to confirm that each equals x.
9. $\left(f^{-1}\right)'(20) = \frac{1}{f'(2)} = 1/5$
11. $\left(f^{-1}\right)'(\sqrt{3}/2) = \frac{1}{f'(\quad /6)} = 1$
13. $\left(f^{-1}\right)'(1/2) = \frac{1}{f'(1)} = -2$
15. $h'(t) = \frac{2}{\sqrt{1-4t^2}}$
17. $g'(x) = \frac{2}{1+4x^2}$
19. $g'(t) = \cos^{-1}(t)\cos(t) - \frac{\sin(t)}{\sqrt{1-t^2}}$
21. $h'(x) = \frac{\sin^{-1}(x)+\cos^{-1}(x)}{\sqrt{1-x^2}\cos^{-1}(x)^2}$
23. $f'(x) = -\frac{1}{\sqrt{1-x^2}}$
25. (a) $f(x) = x$, so $f'(x) = 1$
    (b) $f'(x) = \cos(\sin^{-1}x)\frac{1}{\sqrt{1-x^2}} = 1.$
27. (a) $f(x) = \sqrt{1-x^2}$, so $f'(x) = \frac{-x}{\sqrt{1-x^2}}$
    (b) $f'(x) = \cos(\cos^{-1}x)(\frac{1}{\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$
29. $y = -4(x-\sqrt{3}/4)+ \quad /6$
31. $y = -4/5(x-1)+2$