Eigenvalues and Eigenvectors

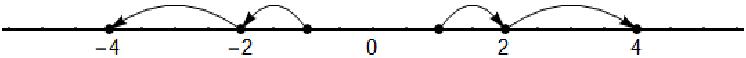
Sean F. Ellermeyer

Kennesaw State University

April 2, 2024

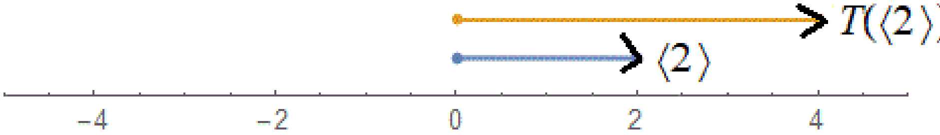
A linear transformation $T : R \rightarrow R$ is a function of the form $T(x) = ax$ where a is a real number constant.

The behavior of linear transformations $T : R \rightarrow R$ is easy to visualize. For example the linear transformation $T(x) = 2x$ maps each point in R to a point that is twice as far from the origin as x and lies on the same side of the origin as x.



The Simple Behavior of Linear Transformations $T : R \longrightarrow R$ (continued)

If we think of the points in R as vectors, then instead of writing $T(x) = 2x$ we can write $T(\mathbf{x}) = 2\mathbf{x}$ and the interpretation is that T maps every vector in $\mathbf{x} \in R$ to a vector that is double the magnitude of $\mathbf{x}$ and points in the same direction as $\mathbf{x}$.



The Simple Behavior of Linear Transformations $T : R \longrightarrow R$ (continued)

To make sure that we understand this interpretation, consider the linear transformation $T : R \rightarrow R$ that is defined by the rule

$$T(\mathbf{x}) = -\frac{1}{3}\mathbf{x}.$$

Fill in the blanks in the statement below and draw pictures of the vectors $\langle 9 \rangle$ and $T(\langle 9 \rangle)$:
T maps each vector in $\mathbf{x} \in R$ to a vector that is _________ the length of $\mathbf{x}$ and points in the _________ direction of $\mathbf{x}$.

Two–Dimensional Analog Example

Consider the linear transformation $T : R^2 \to R^2$ defined by the formula $T(\mathbf{x}) = 2\mathbf{x}$.

First let us be sure that we believe that this is a linear transformation: The more detailed way to write $T(\mathbf{x}) = 2\mathbf{x}$ is

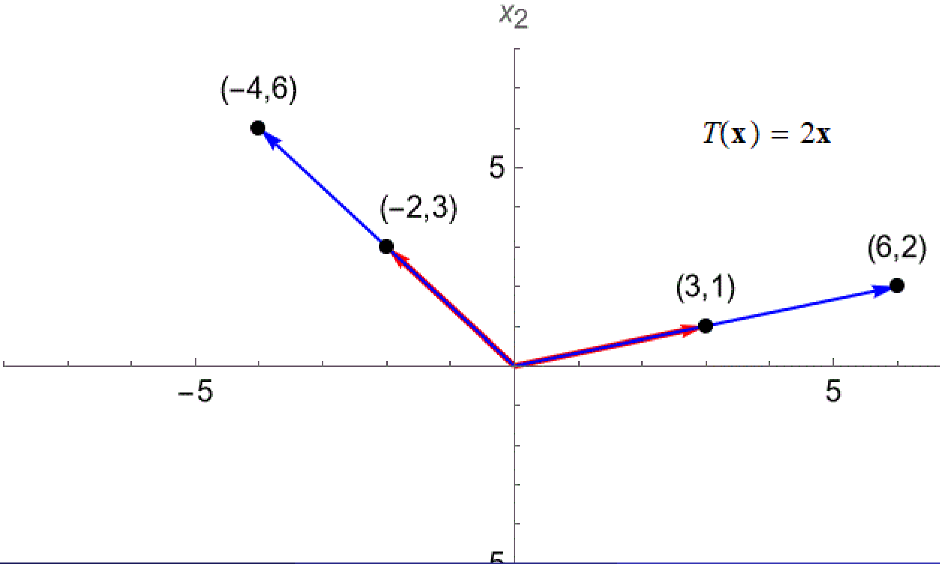
$$T(\langle x_1, x_2 \rangle) = 2\langle x_1, x_2 \rangle = \langle 2x_1, 2x_2 \rangle$$

so we see that this is a linear transformation with coefficient matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I_2.$$

Since $T(\mathbf{x}) = 2\mathbf{x}$ for all $\mathbf{x} \in R^2$, then it is easy to visualize the effect of T. It maps each vector, $\mathbf{x}$, in R^2 to a vector that is double the magnitude of $\mathbf{x}$ and points in the same direction as $\mathbf{x}$.

Picture



The General Idea of Eigenvalues and Eigenvectors

The linear transformations $T : R^n \to R^n$ that have the form $T(\mathbf{x}) = a\mathbf{x}$ where a is a real number constant are the easiest linear transformations to visualize and understand. The transformation $T(\mathbf{x}) = a\mathbf{x}$ maps each vector $\mathbf{x} \in R^n$ to a vector that is $|a|$ times the magnitude of $\mathbf{x}$ and points either in the same direction as $\mathbf{x}$ (if $a > 0$) or in the opposite direction of x (if $a < 0$). (A special case is $a = 0$. In this case $T(\mathbf{x}) = \mathbf{0}_n$ for all $\mathbf{x} \in R^n$.)
The coefficient matrix of $T(\mathbf{x}) = a\mathbf{x}$ is

$$A = \begin{bmatrix} a & 0 & \cdots & 0 & 0 \\ 0 & a & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & a \end{bmatrix} = aI_n.$$

The General Idea of Eigenvalues and Eigenvectors

Of course, most linear transformations are not as simple to understand as the ones of the form $T(\mathbf{x}) = a\mathbf{x}$.

However, given any linear transformation $T : R^n \to R^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$, we can ask the question:

Does there exist any vector $\mathbf{x} \in R^n$ that is mapped by T to a vector that points in the same or the opposite direction as $\mathbf{x}$?

In other words:

Does there exist any vector $\mathbf{x} \in R^n$ with $\mathbf{x} \neq \mathbf{0}_n$ and some scalar λ such that $T(\mathbf{x}) = \lambda\mathbf{x}$?

If so, then we call λ and **eigenvalue** of T and we call $\mathbf{x}$ an **eigenvector** of T that is **associated** with the eigenvalue λ.

Example

Consider the linear transformation $T : R^2 \to R^2$ defined by the rule $T(\mathbf{x}) = A\mathbf{x}$ where the coefficient matrix of T is

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}.$$

Does T have any eigenvalues and eigenvectors?

Example (continued)

Let us inquire as to whether $\lambda = 2$ is an eigenvalue of the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ where the coefficient matrix of T is

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}.$$

To do this we need to consider the equation $T(\mathbf{x}) = 2\mathbf{x}$, which is the same as $A\mathbf{x} = 2\mathbf{x}$, and ask whether there exists a non–zero vector $\mathbf{x} \in R^2$ that satisfies this equation.

Note that it is obviously true that $A\mathbf{0}_2 = 2\mathbf{0}_2$ but we are wondering whether or not $A\mathbf{x} = 2\mathbf{x}$ has any *non–trivial* solutions.

Example (continued)

The equation $A\mathbf{x} = 2\mathbf{x}$ is

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which is the same as the system of equations

$$\begin{aligned} x_1 + 6x_2 &= 2x_1 \\ 5x_1 + 2x_2 &= 2x_2 \end{aligned}$$

or

$$\begin{aligned} -x_1 + 6x_2 &= 0 \\ 5x_1 &= 0. \end{aligned}$$

Example (continued)

It is easily see that the system

$$\begin{aligned} -x_1 + 6x_2 &= 0 \\ 5x_1 &= 0 \end{aligned}$$

has only the trivial solution $x_1 = x_2 = 0$.
This means that there does **not** exist a non–zero vector $\mathbf{x} \in R^2$ such that $T(\mathbf{x}) = 2\mathbf{x}$. Therefore, the scalar $\lambda = 2$ is **not** an eigenvalue of the linear transformation T.

Example (continued)

Let us show that $\lambda = -4$ **is** an eigenvalue of the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ where the coefficient matrix of T is

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}.$$

To do this we need to consider the equation $T(\mathbf{x}) = -4\mathbf{x}$, which is the same as $A\mathbf{x} = -4\mathbf{x}$, and we need to show that this equation has non–trivial solutions.

Example (continued)

The equation $A\mathbf{x} = -4\mathbf{x}$ is

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -4 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which is the same as the system of equations

$$\begin{aligned} x_1 + 6x_2 &= -4x_1 \\ 5x_1 + 2x_2 &= -4x_2 \end{aligned}$$

or

$$\begin{aligned} 5x_1 + 6x_2 &= 0 \\ 5x_1 + 6x_2 &= 0. \end{aligned}$$

Example (continued)

We see that the system

$$5x_1 + 6x_2 = 0$$
$$5x_1 + 6x_2 = 0$$

has infinitely many non–trivial solutions:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{6}{5}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{6}{5} \\ 1 \end{bmatrix}.$$

This shows that $\lambda = -4$ is an eigenvalue of T and that any non–zero scalar multiple of the vector $\mathbf{x} = \left\langle -\frac{6}{5}, 1 \right\rangle$ is an eigenvector of T that is associated with the eigenvalue $\lambda = -4$.

Definition of the Terms Eigenvalue and Eigenvector

Suppose that $T : R^n \to R^n$ is a linear transformation with coefficient matrix A.

We say that a number λ is an **eigenvalue** of T if there is at least one vector $\mathbf{x} \in R^n$ with $\mathbf{x} \neq \mathbf{0}_n$ such that $T(\mathbf{x}) = \lambda \mathbf{x}$.

If λ is an eigenvalue of T and $\mathbf{x}$ is a non–zero vector such that $T(\mathbf{x}) = \lambda \mathbf{x}$, then we say that $\mathbf{x}$ is an **eigenvector** of T that is **associated** with the eigenvalue λ.

Definitions in Terms of Matrices

We know that if $T : R^n \rightarrow R^n$ is a linear transformation, then T is defined by a rule of the form $T\mathbf{x} = A\mathbf{x}$ where A is an $n \times n$ matrix. We thus can refer to the eigenvalues and eigenvectors of an $n \times n$ matrix and make the following definitions:

We say that a number λ is an **eigenvalue** of A if there is at least one vector $\mathbf{x} \in R^n$ with $\mathbf{x} \neq \mathbf{0}_n$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

If λ is an eigenvalue of A and $\mathbf{x}$ is a non–zero vector such that $A\mathbf{x} = \lambda\mathbf{x}$, then we say that $\mathbf{x}$ is an **eigenvector** of A that **associated** with the eigenvalue λ.

How to Check Whether a Certain Scalar is an Eigenvalue

In general, if we are given an $n \times n$ matrix A and we want to check whether or not a certain scalar λ is an eigenvalue of A, we consider the equation $A\mathbf{x} = \lambda\mathbf{x}$ which can be written as $A\mathbf{x} = \lambda\left(I_n\mathbf{x}\right)$ or as the homogeneous equation $\left(A - \lambda I_n\right)\mathbf{x} = \mathbf{0}_n$. We then use the methods we know to determine whether or not this homogeneous equation has a unique solution (which would have to be $\mathbf{x} = \mathbf{0}_n$) or whether it has infinitely many solutions.

If $\left(A - \lambda I_n\right)\mathbf{x} = \mathbf{0}_n$ has infinitely many solutions then λ is an eigenvalue of A and any non–zero vector $\mathbf{x}$ that satisfies $\left(A - \lambda I_n\right)\mathbf{x} = \mathbf{0}_n$ is an eigenvector A associated with the eigenvalue λ. If $\left(A - \lambda I_n\right)\mathbf{x} = \mathbf{0}_n$ has only the trivial solution, then λ is not an eigenvalue of A.

Eigenspaces

Suppose that A is an $n \times n$ matrix and suppose that λ is an eigenvalue of A.

We define the **eigenspace of A associated with the eigenvalue λ**, which we denote by $E(\lambda)$, to be the set of all eigenvectors of A that are associated with λ, with $\mathbf{0}_n$ also thrown in. Thus

$$E(\lambda) = \{\mathbf{x} \in R^n \mid A\mathbf{x} = \lambda\mathbf{x}\}.$$

Remark

We have observed that the equation $A\mathbf{x} = \lambda\mathbf{x}$ can be written as the homogeneous equation

$$(A - \lambda I_n)\,\mathbf{x} = \mathbf{0}_n.$$

Thus instead of writing

$$E(\lambda) = \{\mathbf{x} \in R^n \mid A\mathbf{x} = \lambda\mathbf{x}\}$$

we can write

$$E(\lambda) = \{\mathbf{x} \in R^n \mid (A - \lambda I_n)\,\mathbf{x} = \mathbf{0}_n\}.$$

Another Remark

Observe that the eigenspace

$$E(\lambda) = \{\mathbf{x} \in R^n \mid (A - \lambda I_n)\mathbf{x} = \mathbf{0}_n\}$$

is the null space of the matrix $A - \lambda I_n$, and we already knew that the null space of any $n \times n$ matrix is a subspace of R^n. Thus $E(\lambda)$ is a subspace of R^n.

Example

In a previous example we showed that $T : R^2 \to R^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

has eigenvalue $\lambda = -4$ with corresponding eigenspace

$$E(-4) = \text{Span}\left\{ \begin{bmatrix} 6 \\ -5 \end{bmatrix} \right\}.$$

Show that $\lambda = 7$ is also an eigenvalue of A. Express $E(7)$ in the form $E(7) = \text{Span}\{\_\_\_\_\_\_\}$. Draw pictures of the eigenspaces $E(-4)$ and $E(7)$.

Homework

In Section 5.1, do problems 1–16 (all) and 21–30 (all).