February 14 Math 3260 sec. 51 Spring 2024

Section 2.1: Matrix Operations

Recall the convenient notaton for a matrix A

$$A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Here each column is a vector $\mathbf{a}_j$ in $\mathbb{R}^m$. We'll use the additional convenient notation to refer to A by entries

$$A = [a_{ij}].$$

a_{ij} is the entry in **row** i and **column** j.

Main Diagonal & Diagonal Matrices

The **main diagonal** of a matrix consist of the entries a_{ii}.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{22} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

A **diagonal matrix** is a square matrix, $m = n$, for which all entries **not** on the main diagonal are zero.

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

Matrix Equality

Matrix Equality:

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal provided they are of the same size, $m \times n$, and

$$a_{ij} = b_{ij} \quad \text{for every} \quad i = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n.$$

In this case, we can write

$$A = B.$$

Scalar Multiplication & Matrix Addition

We have two initial operations we can perform on matrices.

Scalar Multiplication:

For $m \times n$ matrix $A = [a_{ij}]$ and scalar c

$$cA = [ca_{ij}].$$

Matrix Addition:

For $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$

$$A + B = [a_{ij} + b_{ij}].$$

Note: The sum of two matrices is only defined if they are of the same size.

Example

Consider the following matrices.

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

Evaluate each expression or state why it fails to exist.

(a) 3B $= \begin{bmatrix} 3(-2) & 3(4) \\ 3(7) & 3(0) \end{bmatrix} = \begin{bmatrix} -6 & 12 \\ 21 & 0 \end{bmatrix}$

Evaluate each expression or state why it fails to exist.

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

(b) A + B $= \begin{bmatrix} 1+(-2) & -3+4 \\ -2+7 & 2+0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 5 & 2 \end{bmatrix}$

(c) C + A C is 2x3, A is 2x2 C+A isn't defined

Zero Matrix

The $m \times n$ **zero matrix** has a zero in each entry. We'll denote this matrix as O (or $O_{m,n}$ if the size is not clear from the context).

Theorem: Algebraic Properties of Scalar Mult. and Matrix Add.

Let A, B, and C be matrices of the same size and r and s be scalars. Then

(i) $A + B = B + A$

(ii) $(A + B) + C = A + (B + C)$

(iii) $A + O = A$

(iv)[a] $A + (-A) = O$

(v) $r(A + B) = rA + rB$

(vi) $(r + s)A = rA + sA$

(vii) $r(sA) = s(rA) = (rs)A$

(viii) $1A = A$

---

[a] The term $-A$ denotes $(-1)A$.

Matrix Multiplication

We know that for any $m \times n$ matrix A, the operation **"multiply vectors in $\mathbb{R}^n$ by A"** defines a linear transformation (from $\mathbb{R}^n$ to $\mathbb{R}^m$).

We wish to define matrix multiplication in such a way as to correspond to **function composition.** Thus if

$$S(\mathbf{x}) = B\mathbf{x}, \quad \text{and} \quad T(\mathbf{v}) = A\mathbf{v},$$

then

$$(T \circ S)(\mathbf{x}) = T(S(\mathbf{x})) = A(B\mathbf{x}) = (AB)\mathbf{x}.$$

Matrix Multiplication

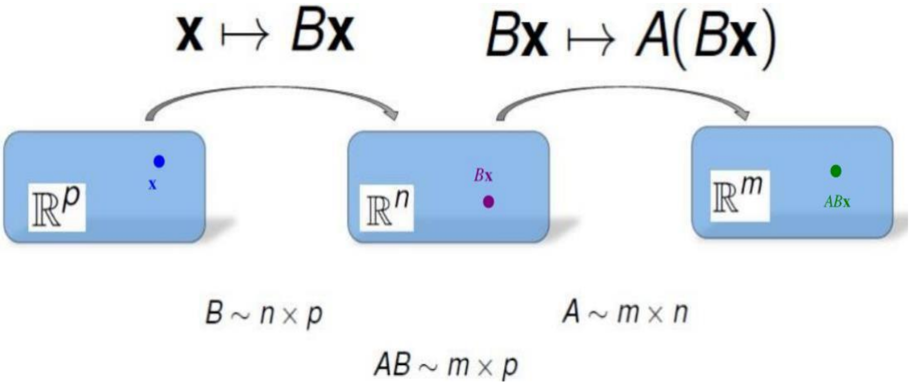

Figure: $\mathbf{x}$ is mapped from $\mathbb{R}^p$ to $B\mathbf{x}$ in $\mathbb{R}^n$. Then $B\mathbf{x}$ in $\mathbb{R}^n$ is mapped to $AB\mathbf{x}$ in $\mathbb{R}^m$. The composition is a mapping $\mathbb{R}^p \to \mathbb{R}^m$. This is only defined if the number of rows of the matrix B is equal to the number of columns of the matrix A.

Matrix Multiplication

$$S : \mathbb{R}^p \longrightarrow \mathbb{R}^n \quad \Longrightarrow \quad B \sim n \times p$$

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \Longrightarrow \quad A \sim m \times n$$

$$T \circ S : \mathbb{R}^p \longrightarrow \mathbb{R}^m \quad \Longrightarrow \quad AB \sim m \times p$$

$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p \Longrightarrow$$

$$A(B\mathbf{x}) = x_1A\mathbf{b}_1 + x_2A\mathbf{b}_2 + \cdots + x_pA\mathbf{b}_p \Longrightarrow$$

$$AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p]$$

The j^{th} column of AB is A times the j^{th} column of B.

Product of Matrices

The product AB is only defined if the number of columns of A (the left matrix) matches the number of rows of B (the right matrix).

$$A\ B$$

$$\color{blue}{m} \times \color{magenta}{n}\ \color{magenta}{n} \times \color{green}{p}$$

$$\downarrow$$

$$\color{blue}{m} \times \color{green}{p}$$

$$AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p]$$

Example

Compute the product AB where

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

AB: 2×2, 2×3 ✓ → 2×3

$$\vec{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} 0 \\ -4 \end{bmatrix}, \quad \vec{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$A\vec{b}_1 = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$A\vec{b}_2 = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ -4 \end{bmatrix} = 0\begin{bmatrix} 1 \\ -2 \end{bmatrix} - 4\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \end{bmatrix}$$

$$A\vec{b}_3 = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 6\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -16 \\ 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 12 & -16 \\ -2 & -8 & 8 \end{bmatrix}$$

Row-Column Rule for Computing the Matrix Product

If $AB = C = [c_{ij}]$, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

(The ij^{th} entry of the product is the *dot product* of i^{th} row of A with the j^{th} column of B.)

For example, if A is 2×2 and B is 2×3, then $n = 2$. The entry in row 2 column 3 of AB would be

$$c_{23} = \sum_{k=1}^{2} a_{2k} b_{k3} = a_{21} b_{13} + a_{22} b_{23}.$$

Example

For example: $AB = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix} =$

A B
2×2 2×3
↓
2×3

$$\begin{bmatrix} -1 & 12 & -16 \\ -2 & -8 & 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 12 & -16 \\ -2 & -8 & 8 \end{bmatrix}$$

Theorem: Properties of the Matrix Product

Let A be an $m \times n$ matrix. Let r be a scalar and B and C be matrices for which the indicated sums and products are defined. Then

(i) $A(BC) = (AB)C$

(ii) $A(B + C) = AB + AC$

(iii) $(B + C)A = BA + CA$

(iv) $r(AB) = (rA)B = A(rB)$, and

(v) $I_m A = A = AI_n$

Critical Remarks

Caveats

1. Matrix multiplication **does not commute**! That is, in general $AB \neq BA$. In fact, the validity of AB does not even imply that BA is defined.

2. The zero product property **does not** hold! That is, if $AB = O$, one **cannot** conclude that one of the matrices A or B is a zero matrix.

3. There is **No** *cancelation law*. That is, $AB = CB$ **does not** imply that A and C are equal.

Compute AB and BA where $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$.

AB: 2×2 ✓ 2×2 → 2×2

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & 6 \end{bmatrix}$$

BA: 2×2 2×2 → 2×2

$$BA = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 11 \\ -1 & 4 \end{bmatrix}$$

Both AB and BA are defined but $AB \neq BA$.

Compute the products AB, CB, and BB where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,

$B = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$CB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

Note $AB = CB$ but $A \neq C$

$$BB = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$BB = 0$ but $B \neq 0$

B^2

Matrix Powers

Positive Integer Powers:

If A is square—meaning A is an $n \times n$ matrix for some $n \geq 2$, then the product AA is defined. For positive integer k, we'll define

$$A^k = AA^{k-1}.$$

Zero Power: We define $A^0 = I_n$, where I_n is the $n \times n$ identity matrix.

Transpose

Definition: Matrix Transpose

Let $A = [a_{ij}]$ be an $m \times n$ matrix. The **transpose** of A is the $n \times m$ matrix denoted and defined by

$$A^T = [a_{ji}].$$

For example, if

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, \quad \text{then} \quad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}.$$

Example

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

Compute A^T, B^T, the transpose of the product $(AB)^T$, and the product $B^T A^T$.

We already computed $AB = \begin{bmatrix} -1 & 12 & -16 \\ -2 & -8 & 8 \end{bmatrix}$ in a previous example.

$$A^T = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix} \qquad B^T = \begin{bmatrix} 2 & 1 \\ 0 & -4 \\ 2 & 6 \end{bmatrix} \qquad (AB)^T = \begin{bmatrix} -1 & -2 \\ 12 & -8 \\ -16 & 8 \end{bmatrix}$$

$B^T A^T$

$3\times 2 \quad 2\times 2$

$\downarrow$

3×2

$$B^T A^T = \begin{bmatrix} 2 & 1 \\ 0 & -4 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 \\ 12 & -8 \\ -16 & 9 \end{bmatrix}$$

Properties-Matrix Transposition

Theorem

Let A and B be matrices such that the appropriate sums and products are defined, and let r be a scalar. Then

(i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$

(iii) $(rA)^T = rA^T$

(iv) $(AB)^T = B^T A^T$