February 16 Math 3260 sec. 52 Spring 2024

Section 2.2: Inverse of a Matrix

Consider the scalar equation $ax = b$. Provided $a \neq 0$, we can solve this explicity

$$x = a^{-1}b$$

where a^{-1} is the unique number such that $aa^{-1} = a^{-1}a = 1$.

If A is an $n \times n$ matrix, we seek an analog A^{-1} that satisfies the condition

$$A^{-1}A = AA^{-1} = I_n.$$

- If such matrix A^{-1} exists, we'll say that A is **nonsingular** or **invertible.**
- Otherwise, we'll say that A is **singular**.

2×2 case

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is singular.

Determinant

The quantity $ad-bc$ is called the **determinant** of A and may be denoted in several ways

$$\det(A) \;=\; |A| \;=\; \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Find the inverse if possible

(a) $A = \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}$

$\det(A) = 3(5) - (-1)(2) = 17$

$\det(A) \neq 0 \Rightarrow A^{-1}$ exists.

$$A^{-1} = \frac{1}{17}\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$$

Check:

$$A^{-1}A = \frac{1}{17}\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Find the inverse if possible

(b) $A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$

$\det(A) = 3(4) - 6(2) = 12 - 12 = 0$

A is singular, a.k.a. not invertible.

Theorem

Theorem

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof: Suppose A is invertible with inverse A^{-1}. Let $\vec{x} = A^{-1}\vec{b}$ and show that $\vec{x}$ solves $A\vec{x} = \vec{b}$.

$$\vec{x} = A^{-1}\vec{b}$$

Multiply each side on the left by A.

$$A\vec{x} = A(A^{-1}\vec{b})$$

$$A\vec{x} = (AA^{-1})\vec{b}$$

$$A\vec{x} = I\vec{b} = \vec{b} \Rightarrow A^{-1}\vec{b} \text{ solves } A\vec{x} = \vec{b}.$$

Now, consider $A\vec{x} = \vec{b}$. Multiply on the left by A^{-1}.

$$A\vec{x} = \vec{b}$$

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b} \Rightarrow A^{-1}\vec{b} \text{ is the only solution.}$$

Hence $A^{-1}\vec{b}$ is the unique solution to

$$A\vec{x} = \vec{b}.$$

Example

Use a matrix inverse to solve the system.

$$\begin{array}{rcrcr} 3x_1 & + & 2x_2 & = & -1 \\ -x_1 & + & 5x_2 & = & 4 \end{array}$$

restate this as $A\vec{x} = \vec{b}$.

$$\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix},$$

call $\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix} = A$ and $\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \vec{b}$.

From before $A^{-1} = \frac{1}{17} \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$.

By our theorem, $\vec{x} = A^{-1}\vec{b}$

$$\vec{x} = \frac{1}{17}\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} -13 \\ 11 \end{bmatrix} = \begin{bmatrix} -13/17 \\ 11/17 \end{bmatrix}$$

$$x_1 = \frac{-13}{17}, \quad x_2 = \frac{11}{17}$$

Inverses, Products, & Transposes

Theorem

(i) If A is invertible, then A^{-1} is also invertible and

$$\left(A^{-1}\right)^{-1} = A.$$

(ii) If A and B are invertible $n \times n$ matrices, then the product AB is also invertible[a] with

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(iii) If A is invertible, then so is A^T. Moreover

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T.$$

$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$

[a] This can generalize to the product of k invertible matrices.

Elementary Matrices

Definition:

An **elementary** matrix is a square matrix obtained from the identity by performing one elementary row operation.

Examples[1]:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$3R_2 \to R_2$ $\qquad$ $2R_1 + R_3 \to R_3$ $\qquad$ $R_1 \leftrightarrow R_2$

[1] There's nothing standard about the subscripts used here, although using E to denote an elementary matrix is common.

Action of Elementary Matrices

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, and compute the following products

$$E_1A, \quad E_2A, \quad \text{and} \quad E_3A.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{bmatrix}.$$

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

$$E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= \begin{bmatrix} a & b & c \\ d & e & f \\ 2a+g & 2b+h & 2c+i \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$E_3 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Remarks

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1. Elementary row operations can be equated with matrix multiplication (multiply on the left by an elementary matrix),
2. Each elementary matrix is invertible where the inverse *undoes* the row operation,
3. Reduction to rref is a sequence of row operations, so it is a sequence of matrix multiplications

$$\mathrm{rref}(A) = E_k \cdots E_2 E_1 A.$$