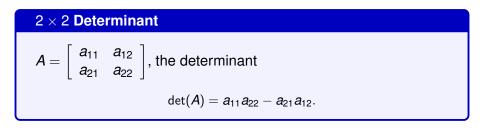
February 26 Math 3260 sec. 51 Spring 2024

Section 3.1: Introduction to Determinants

Here, we want to extend the concept of **determinant** to all $n \times n$ matrices and do it in such a way that for any square matrix A,

A is nonsinguar if and only if $det(A) \neq 0$.



Determinant: 3×3 Matrix

$$3 \times 3 \text{ Determinant}$$

For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the determinant
 $\det(A) = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

Remark: Note that this is a sum or difference of the entries in the top row where each is multiplied by the determinant of a 2×2 matrix obtained by eliminating the row and column of that entry. These determinants of *sub-matrices* have a name. They're called **minors**.

Minors & Cofactors

Some Notation

Let $n \ge 2$. For an $n \times n$ matrix A, let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i^{th} row and the j^{th} column of A.

For example, if

$$A = \begin{bmatrix} -1 & 3 & 2 & 0 \\ 4 & 4 & 0 & -3 \\ -2 & 1 & 7 & 2 \\ 3 & 0 & -1 & 6 \end{bmatrix} \text{ then } A_{23} = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix}$$

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Minors & Cofactors

Suppose *A* is an $n \times n$ matrix for some $n \ge 2$.

Definition: Minor

The *i*, *j*th **minor** of the $n \times n$ matrix *A* is the number

$$M_{ij} = \det(A_{ij}).$$



Definition: Cofactor

Let *A* be an $n \times n$ matrix with $n \ge 2$. The *i*, *j*th **cofactor** of *A* is the number

$$C_{ij}=(-1)^{i+j}M_{ij}.$$

Remark: Minors can be computed by removing more than one row and column (as long as the resulting matrix is still square). Some people would one call what I've defined here a **first minor**.

Minors & Cofactors

Find the three minors M_{11} , M_{12} , M_{13} and find the 3 cofactors C_{11} , C_{12} , C_{13} of the matrix

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$$C_{11} = (-1)^{141} M_{11} = M_{11} = a_{22} a_{33} - a_{32} a_{23}$$

$$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 M_{12} = -M_{12}$$

= - ($a_{21} a_{33} - a_{31} a_{23}$)

$$C_{13} = (-D M_{13} = (-U M_{13} = M_{13} = A_{21} A_{32} - A_{31} A_{22})$$

Observation:

Recall that the determinant of the 3 × 3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ was given by

$$\det(A) = a_{11}\det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Cofactor Expansion

Note that we can write

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

An expression of this form is called a *cofactor expansion*.

The Determinant

Definition: Determinant

For $n \ge 2$, the **determinant** of the $n \times n$ matrix $A = [a_{ij}]$ is the number

$$det(A) = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j}M_{1j}$$

Remark: Note that this definition defines determinants iteratively via the minors. The determinant of a 3×3 matrix is given in terms of the determinants of three 2×2 matrices. The determinant of a 4×4 matrix is given in terms of the determinants of four 3×3 matrices, and so forth.

Example Evaluate det(*A*).

$$A = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix} \qquad dek(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$C_{11} = (-1)^{1+1} dek(\begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}) = 6$$
$$C_{12} = (-1)^{1+2} dek(\begin{bmatrix} -2 & 2 \\ 3 & 6 \end{bmatrix}) = (-1)(-12 - 6) = 18$$
$$C_{13} = (-1)^{1+3} dek(\begin{bmatrix} -2 & 1 \\ 3 & 6 \end{bmatrix}) = 0 - 3 = -3$$
$$dek(A) = (-1)(6) + 3(18) + (6)(-3) = 48$$

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Example

Find all values of x such that det(A) = 0.

$$A = \begin{bmatrix} 3-x & 2 & 1 \\ 0 & 2-x & 4 \\ 0 & 3 & 1-x \end{bmatrix} \quad d(A) = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$det(A) = (3-x)(-1) \begin{vmatrix} 2-x & y \\ 3 & 1-x \end{vmatrix} + 2(-1) \begin{vmatrix} 0 & y \\ 0 & 1-x \end{vmatrix} + 1(-1) \begin{vmatrix} 1+3 \\ 0 & 2-x \end{vmatrix}$$

$$= (3-x) \left((z-x)(1-x) - 1z \right) - z \cdot 0 + 1 \cdot 0$$

$$= (3 - \chi) (2 - 3\chi + \chi^{2} - (Z))$$

$$= (3 - \chi) (\chi^{2} - 3\chi - ID) \quad (D + (B) + (Z) + (Z) + Z) = 0.00$$
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$$det(A) = (3-x)(x^2 - 3x - 10)$$
$$= (3-x)(x - 5)(x + 2)$$

$$det(A) = 0$$
 if $3 - x = 0$, $x - 5 = 0$; or $x + 2 = 0$.
 $det(A) = 0$ if $x = 3$, $x = 5$, or $x = -2$

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General Cofactor Expansions

Theorem

The determinant of an $n \times n$ matrix can be computed by cofactor expansion across any row or down any column.

We can fix any row *i* of a matrix *A* and then

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

Or, we can fix any column *j* of a matrix *A* and then

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

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Example

Evaluate det(A).

 $A = \left| \begin{array}{rrrrr} -1 & 3 & 4 & 0 \\ 0 & 0 & -3 & 0 \\ -2 & 1 & 2 & 2 \\ 3 & 0 & -1 & 6 \end{array} \right|$

Across row Z

$$d_{xt}(A) = a_{x_1}^{0} (z_1 + a_{z_2}^{2}) (z_2 + a_{z_3}^{2}) (z_3 + a_{z_3}^{2}) (z_4)$$

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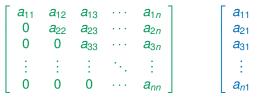
= 144

Triangular Matrices

Definition:

The $n \times n$ matrix $A = [a_{ij}]$ is said to be **upper triangular** if $a_{ij} = 0$ for all i > j.

It is said to be **lower triangular** if $a_{ij} = 0$ for all j > i. A matrix that is both upper and lower triangular is a **diagonal** matrix.



Upper Triangular

a ₁₁	0	0		0]
a ₂₁	a 22	0		0
a ₃₁	a 32	a 33	• • •	0
1	÷	- 1	γ_{i_1}	÷
<i>a</i> _{n1}	an2	a _{n3}	•••	ann

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Determinant of Triangular Matrix

Theorem:

For $n \ge 2$, the determinant of an $n \times n$ triangular matrix is the product of its diagonal entries. (i.e. if $A = [a_{ij}]$ is triangular, then $det(A) = a_{11}a_{22}\cdots a_{nn}$.)

Example: Evaluate det(*A*)

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Example

Evaluate det(A)

$$A = \begin{bmatrix} -1 & 3 & 4 & 0 & 2 \\ 0 & 2 & -3 & 0 & -4 \\ 0 & 0 & 3 & 7 & 5 \\ 0 & 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

$$det(A) = (-1)(z)(3)(-u)(6) = 6(zu)$$
$$= 144$$

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Section 3.2: Properties of Determinants

Theorem:

Let *A* be an $n \times n$ matrix, and suppose the matrix *B* is obtained from *A* by performing a single elementary row operation^{*a*}. Then

(i) If *B* is obtained by adding a multiple of a row of *A* to another row of *A* (row replacement), then

 $\det(B) = \det(A).$

(ii) If *B* is obtained from *A* by swapping any pair of rows (row swap) , then

 $\det(B) = -\det(A).$

(iii) If *B* is obtained from *A* by scaling any row by the constant *k* (scaling), then

 $\det(B) = k \det(A).$

^aIf "row" is replaced by "column" in any of the operations, the conclusions still follow.