February 26 Math 3260 sec. 51 Spring 2024

Section 3.1: Introduction to Determinants

Here, we want to extend the concept of **determinant** to all $n \times n$ matrices and do it in such a way that for any square matrix A,

A is nonsinguar if and only if $\det(A) \neq 0$.

2×2 Determinant

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the determinant

$$\det(A) = a_{11}a_{22} - a_{21}a_{12}.$$

Determinant: 3×3 Matrix

3×3 Determinant

For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the determinant

$$\det(A) = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Remark: Note that this is a sum or difference of the entries in the top row where each is multiplied by the determinant of a 2×2 matrix obtained by eliminating the row and column of that entry. These determinants of *sub-matrices* have a name. They're called **minors**.

Minors & Cofactors

Some Notation

Let $n \geq 2$. For an $n \times n$ matrix A, let A_{ij} denote the $(n-1)\times(n-1)$ matrix obtained from A by deleting the i^{th} row and the j^{th} column of A.

For example, if

$$A = \begin{bmatrix} -1 & 3 & 2 & 0 \\ 4 & 4 & 0 & -3 \\ -2 & 1 & 7 & 2 \\ 3 & 0 & -1 & 6 \end{bmatrix} \quad \text{then} \quad A_{23} = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix}.$$

Minors & Cofactors

Suppose A is an $n \times n$ matrix for some $n \geq 2$.

Definition: Minor

The i, j^{th} **minor** of the $n \times n$ matrix A is the number

$$M_{ij} = \det(A_{ij}).$$

minors are numbers

Definition: Cofactor

Let A be an $n \times n$ matrix with $n \geq 2$. The i, j^{th} **cofactor** of A is the number

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

Remark: Minors can be computed by removing more than one row and column (as long as the resulting matrix is still square). Some people would call what I've defined here a **first minor**.

Minors & Cofactors

Find the three minors M_{11}, M_{12}, M_{13} and find the 3 cofactors C_{11}, C_{12}, C_{13} of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

$$\boxed{M_{11} = \det(A_{11}) = a_{22}a_{33} - a_{23}a_{32}}$$

$$A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$\boxed{M_{12} = \det(A_{12}) = a_{21}a_{33} - a_{31}a_{23}}$$

$$A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$\boxed{M_{13} = \det(A_{13}) = a_{21}a_{32} - a_{31}a_{22}}$$

$$A_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = a_{22}a_{33} - a_{32}a_{23}$$

$$C_{12} = (-1)^{1+2} M_{12} = (-1)^{3} M_{12} = -M_{12}$$

$$= -\left(a_{21}a_{33} - a_{31}a_{23}\right)$$

$$C_{13} = (-1)^{1+3} M_{13} = (-1)^{4} M_{13} = M_{13} = a_{21}a_{32} - a_{31}a_{22}$$

Observation:

Recall that the determinant of the 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

was given by

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Cofactor Expansion

Note that we can write

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

An expression of this form is called a *cofactor expansion*.

The Determinant

Definition: Determinant

For $n \geq 2$, the **determinant** of the $n \times n$ matrix $A = [a_{ij}]$ is the number

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j} \end{aligned}$$

Remark: Note that this definition defines determinants iteratively via the minors. The determinant of a 3×3 matrix is given in terms of the determinants of three 2×2 matrices. The determinant of a 4×4 matrix is given in terms of the determinants of four 3×3 matrices, and so forth.

Example

Evaluate $\det(A)$.

$$A = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix}$$

$\det(A) = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$

$C_{11} = (-1)^{1+1} \det\left(\begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}\right) = 6$

$C_{12} = (-1)^{1+2} \det\left(\begin{bmatrix} -2 & 2 \\ 3 & 6 \end{bmatrix}\right) = (-1)(-12-6) = 18$

$C_{13} = (-1)^{1+3} \det\left(\begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix}\right) = 0-3 = -3$

$\det(A) = (-1)(6) + 3(18) + (0)(-3) = 48$

Example

Find all values of x such that $\det(A) = 0$.

$$A = \begin{bmatrix} 3-x & 2 & 1 \\ 0 & 2-x & 4 \\ 0 & 3 & 1-x \end{bmatrix}$$

$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$$\det(A) = (3-x)(-1)^{1+1}\begin{vmatrix} 2-x & 4 \\ 3 & 1-x \end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix} 0 & 4 \\ 0 & 1-x \end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix} 0 & 2-x \\ 0 & 3 \end{vmatrix}$$

$$= (3-x)\left((2-x)(1-x) - 12\right) - 2 \cdot 0 + 1 \cdot 0$$

$$= (3-x)\left(2 - 3x + x^2 - 12\right)$$

$$= (3-x)\left(x^2 - 3x - 10\right)$$

$$\det(A) = (3-x)(x^2-3x-10)$$

$$= (3-x)(x-5)(x+2)$$

$\det(A) = 0$ if $3-x=0$, $x-5=0$, or $x+2=0$.

$\det(A) = 0$ if $x=3$, $x=5$, or $x=-2$

General Cofactor Expansions

Theorem

The determinant of an $n \times n$ matrix can be computed by cofactor expansion across any row or down any column.

We can fix any row i of a matrix A and then

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

Or, we can fix any column j of a matrix A and then

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

Example

Evaluate $\det(A)$.

$$A = \begin{bmatrix} -1 & 3 & 4 & 0 \\ 0 & 0 & -3 & 0 \\ -2 & 1 & 2 & 2 \\ 3 & 0 & -1 & 6 \end{bmatrix}$$

Across row 2

$$\det(A) = \underbrace{a_{21}}_{=0} C_{21} + \underbrace{a_{22}}_{=0} C_{22} + a_{23} C_{23} + \underbrace{a_{24}}_{=0} C_{24}$$

$$\det(A) = -3\,(-1)^{2+3} \begin{vmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{vmatrix} = -3\,(-1)\,48$$

from previous example

$= 144$

Triangular Matrices

Definition:

The $n \times n$ matrix $A = [a_{ij}]$ is said to be **upper triangular** if $a_{ij} = 0$ for all $i > j$.

It is said to be **lower triangular** if $a_{ij} = 0$ for all $j > i$. A matrix that is both upper and lower triangular is a **diagonal** matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Upper Triangular **Lower Triangular**

Determinant of Triangular Matrix

Theorem:

For $n \geq 2$, the determinant of an $n \times n$ triangular matrix is the product of its diagonal entries. (i.e. if $A = [a_{ij}]$ is triangular, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$.)

Example: Evaluate $\det(A)$

$$A = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 4 & 2 & 2 & 2 \end{bmatrix}$$

$\det(A) = 7 \cdot 6 \cdot 2 \cdot 2$

$= 42 \cdot 4 \quad = 168$

Example

Evaluate $\det(A)$

$$A = \begin{bmatrix} -1 & 3 & 4 & 0 & 2 \\ 0 & 2 & -3 & 0 & -4 \\ 0 & 0 & 3 & 7 & 5 \\ 0 & 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

$$\det(A) = (-1)(2)(3)(-4)(6) = 6(24)$$

$$= 144$$

Section 3.2: Properties of Determinants

Theorem:

Let A be an $n \times n$ matrix, and suppose the matrix B is obtained from A by performing a single elementary row operation[a]. Then

(i) If B is obtained by adding a multiple of a row of A to another row of A (row replacement), then

$$\det(B) = \det(A).$$

(ii) If B is obtained from A by swapping any pair of rows (row swap) , then

$$\det(B) = -\det(A).$$

(iii) If B is obtained from A by scaling any row by the constant k (scaling), then

$$\det(B) = k\det(A).$$

---

[a] If "row" is replaced by "column" in any of the operations, the conclusions still follow.