February 9 Math 3260 sec. 51 Spring 2024

Section 1.9: The Matrix for a Linear Transformation

Recall Linear Transformation

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** provided for every vector $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and every scalar c

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad \text{and}$$

$$T(c\mathbf{u}) = cT(\mathbf{u}).$$

Two Remarks

1. Any mapping defined by matrix multiplication, $\mathbf{x} \mapsto A\mathbf{x}$, is a linear transformation.
2. Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be realized in terms of matrix multiplication.

Elementary Vectors

Definition: Elementary Vectors

We'll use the notation $\mathbf{e}_i$ to denote the vector in $\mathbb{R}^n$ having a 1 in the i^{th} position and zero everywhere else. The vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are called **elementary** vectors.

For example, the elementary vectors in $\mathbb{R}^2$ are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The elementary vectors in $\mathbb{R}^3$ are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Elementary Vectors

Remark:

In general, the elementary vectors are the columns of the $n \times n$ identity matrix.

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$I_n = [\mathbf{e}_1 \ \mathbf{e}_2 \ \cdots \ \mathbf{e}_n]$$

Matrix of Linear Transformation: an Example

Suppose $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ is a linear transformation, and that

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}.$$

Use the fact that T is linear, and the fact that for each $\mathbf{x}$ in $\mathbb{R}^2$ we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$$

to find a matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^2.$$

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}$$

For $\vec{x}$ in $\mathbb{R}^2$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2$

$$T(\vec{x}) = T(x_1 \vec{e}_1 + x_2 \vec{e}_2)$$

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

$$= x_1 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -2 & -1 \\ 4 & 6 \end{bmatrix}}_{\text{call this } A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then $T(\vec{x}) = A\vec{x}$.

Standard Matrix of a Linear Transformation

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^n.$$

Moreover, the j^{th} column of the matrix A is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} column of the $n \times n$ identity matrix I_n. That is,

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)]\,.$$

The matrix A is called the **standard matrix** for the linear transformation T.

Example

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the scaling transformation (contraction or dilation for $r > 0$) defined by

$$T(\mathbf{x}) = r\mathbf{x}, \quad \text{for positive scalar } r.$$

Find the standard matrix for T.

We need $T(\vec{e}_1)$ and $T(\vec{e}_2)$ for $\vec{e}_1, \vec{e}_2$ in $\mathbb{R}^2$.

$$T(\vec{e}_1) = r\vec{e}_1 = r\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

$$T(\vec{e}_2) = r\vec{e}_2 = r\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

The standard matrix

$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

A Shear Transformation on $\mathbb{R}^2$

Find the standard matrix for the linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$ that maps $\mathbf{e}_2$ to $\mathbf{e}_2 - \frac{1}{2}\mathbf{e}_1$ and leaves $\mathbf{e}_1$ unchanged.

Call the transformation S.

$$S(\vec{e}_1) = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$S(\vec{e}_2) = \vec{e}_2 - \frac{1}{2}\vec{e}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

The standard matrix

$$A = \begin{bmatrix} 1 & -1/2 \\ 0 & 1 \end{bmatrix}.$$

A Shear Transformation on $\mathbb{R}^2$

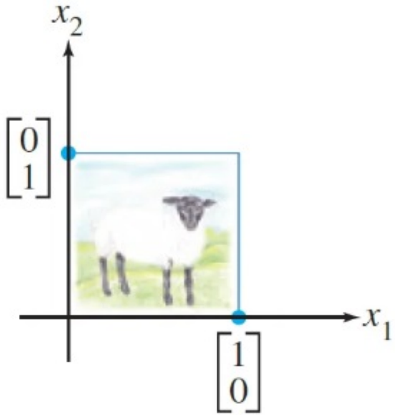
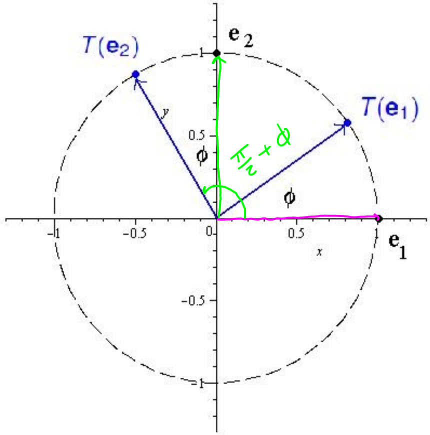


Figure: The unit square under the transformation $\mathbf{x} \mapsto \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \mathbf{x}$.

A Rotation on $\mathbb{R}^2$

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the rotation transformation that rotates each point in $\mathbb{R}^2$ counter clockwise about the origin through an angle ϕ. Find the standard matrix for T.



Using some basic trigonometry, the points on the unit circle

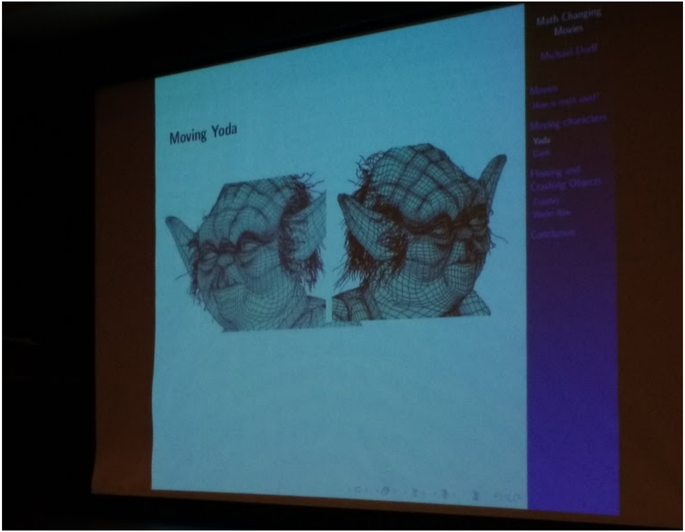
$$T(\mathbf{e}_1) = (\cos\phi, \sin\phi)$$

$$T(\mathbf{e}_2) = (\cos(90^\circ + \phi), \sin(90^\circ + \phi))$$

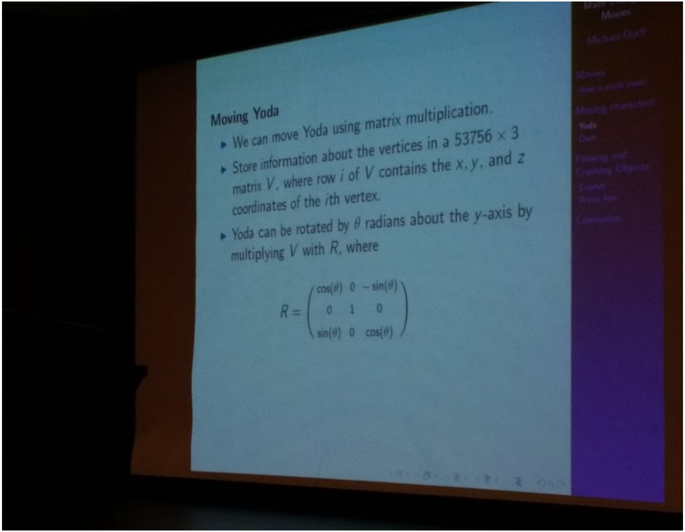
$$= (-\sin\phi, \cos\phi)$$

$$\text{So } A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

Rotation in Animation



Rotation in Animation



Rotation in Curve Generation

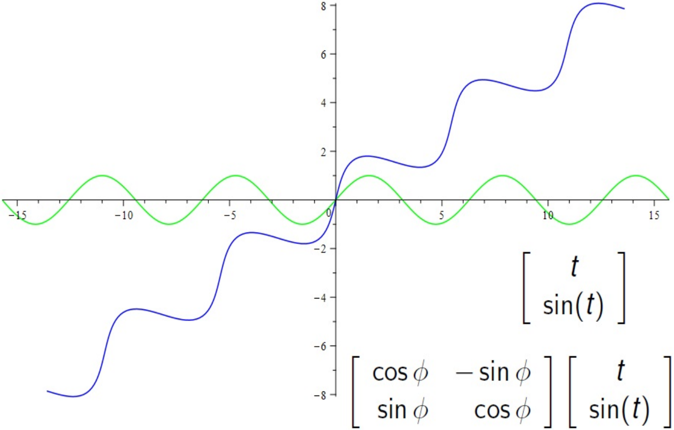


Figure: The curve $y = \sin(x)$ plotted as a vector valued function along with a version rotated through and angle $\phi = \frac{\pi}{6}$.

Onto and One to One

Definition

A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$—i.e. if the range of T is all of the codomain.

Definition

A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **one to one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of **at most one** $\mathbf{x}$ in $\mathbb{R}^n$.

Some Theorems about *Onto* and *One to One*

Theorem:

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is one to one if and only if the homogeneous equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem:

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then

(i) T is onto if and only if the columns of A span $\mathbb{R}^m$, and

(ii) T is one to one if and only if the columns of A are linearly independent.

Remarks

Suppose $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation and A is the standard matrix for T.

- If T is **onto**, then
  - the range of T is $\mathbb{R}^m$,
  - the equation $T(\mathbf{x}) = \mathbf{b}$ is always solvable,
  - the system $A\mathbf{x} = \mathbf{b}$ is always consistent.
- If T is **one to one**, then
  - $T(\mathbf{x}) = T(\mathbf{y})$ implies that $\mathbf{x} = \mathbf{y}$,
  - $A\mathbf{x} = \mathbf{0}$ has no free variables.

Example

Consider the linear transformation $\begin{array}{c} T : \mathbb{R}^3 \to \mathbb{R}^2 \\ (x_1, x_2, x_3) \mapsto (x_3, x_1 + x_2) \end{array}$

Determine the set of all preimages[1] of $\mathbf{0}$. State the solution as a span.

$\vec{x}$ is a preimage of $\vec{0}$ if $T(\vec{x}) = \vec{0}$

We can use the standard matrix A.

$A = [T(\vec{e}_1) \; T(\vec{e}_2) \; T(\vec{e}_3)]$.

$T(\vec{e}_1) = T(1,0,0) = (0, 1+0) = (0,1)$

$T(\vec{e}_2) = T(0,1,0) = (0, 0+1) = (0,1)$

[1] This actually has a special name. The set of all preimages of the zero vector is called the *kernel* of T.

$T(\vec{e}_3) = T(0,0,1) = (1, 0+0) = (1,0)$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

For $A\vec{x} = \vec{0}$, $A \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$x_1 = -x_2$
x_2 - free
$x_3 = 0$

$$\vec{x} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

The set of preimages of $\vec{0}$ is

$$\text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$