January 10 Math 3260 sec. 51 Spring 2024

Section 1.1: Systems of Linear Equations

Consider a generic linear system such as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & \vdots & & \vdots & \cdots & \vdots & & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array} \tag{1}$$

We defined

- a linear equation and a linear system,
- solution and solution set for a linear system, and
- equivalent systems.

The Geometry of 2 Equations with 2 Variables

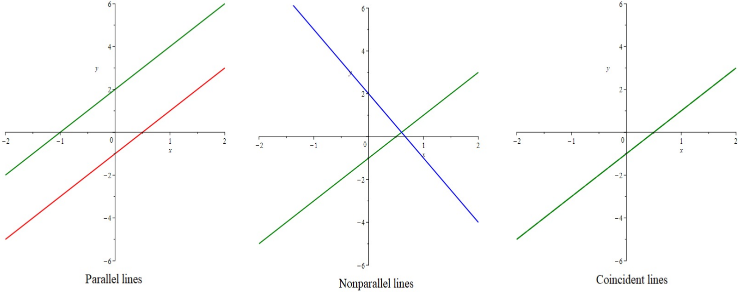

Figure: The geometry of a system of two equations in two variables is easy to see.

> While the geometry is more complex—or even beyond visible representation—the basic trichotomy generalizes to all linear, algebraic systems.

Theorem

Theorem

For a linear system, exactly one of the following holds. The system has

i no solution, or

ii exactly one solution, or

iii infinitely many solutions.

A system is called **inconsistent** if it does not have any solutions (case i), and it's called **consistent** if it has any solution(s) (cases ii & iii).

Note: This theorem speaks to those two big questions:

- Existence: Is there a solution/does a solution exist?
- Uniqueness: Is there a unique solution or multiple solutions?

3 Equations in 3 Variables

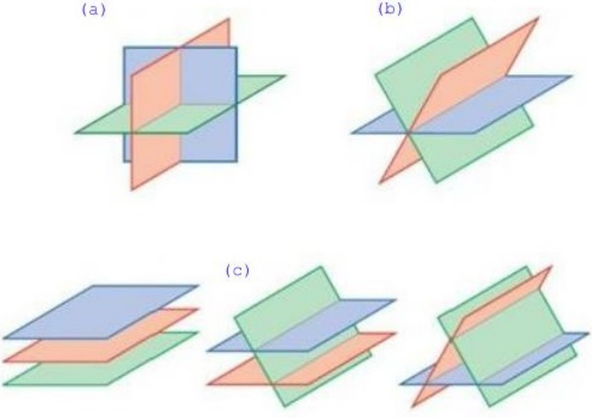

Figure: The graph of $ax + by + cz = d$ is a plane. Three may (a) intersect in a single point, (b) intersect in infinitely many points, or (c) not intersect in various ways.

Matrices

Definition: Matrix

A **matrix** is a rectangular array of numbers. The size (or dimension) of a matrix is written like $m \times n$ (read "m by n") where m is the number of rows and n is the number of columns of the matrix.

Examples:

$$\begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & 1 & 13 & -4 \\ 12 & -3 & 2 & -2 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 0 \\ 4 & 4 \\ 3 & -5 \end{bmatrix}$$

3×4 $\qquad$ 3×2

Linear System & Matrices

Given any linear system of equations, we can associate two matrices with the system. These are the **coefficient** matrix and the **augmented** matrix.

Example:
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

Before we start to set up these matrices, we write our system in the form shown above. Note that all variables are on the left side, and like variables have the same order in each equation (they are aligned vertically).

Linear System: Coefficient Matrix

The **coefficient** matrix has one row for each equation and one column for each variable. The entries are the coefficients of the variables in our system.

m = # of equations

$$\text{Example:}\quad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

n = # of variables

Here, 3×3

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Linear System: Augmented Matrix

The **augmented** matrix has one row for each equation, one column for each variable, and one extra, right most column. The entries in the first columns match the coefficient matrix, and the right most column has the numbers from the right hand side of each equation.

Example:
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

$m = \#$ of equations

$n = \#$ of variable $+1$

Here, $m=3$

$n = 3+1$

so 3×4

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

a delimiter may go here

straight or dashed line

Legitimate Operations for Solving a System

There are three operations that we can perform on a system of equations that result in an **equivalent** system. We can use these operations to *eliminate* variables. We can

- **swap** the order of any two equations,
- **scale** an equation by multiplying it by any **nonzero** number, and
- **replace** an equation with the sum[1] of itself and a nonzero multiple of any other equation.

We will use some standard notation for these operations. Let's call the first equation E_1, the second equation E_2 and so forth.

[1] Adding equations means adding like variables.

Swap

To indicate that we are swapping equations E_i and E_j, we'll write

$$E_i \leftrightarrow E_j$$

For example

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \quad \boxed{E_1 \leftrightarrow E_3} \quad \begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 6 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & 2x_2 & - & x_3 & = & -4 \end{array}$$

Scale

To indicate that we are scaling equation E_i by the nonzero factor k, we'll write

$$kE_i \to E_i$$

For example

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \quad \boxed{-2E_3 \to E_3} \quad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ -2x_1 & - & 2x_2 & - & 2x_3 & = & -12 \end{array}$$

Replace

To indicate that we are replacing equation E_j with the sum of itself adn k times equation E_i, we'll write

$$kE_i + E_j \to E_j$$

For example

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \quad \boxed{-2E_1 + E_2 \to E_2} \quad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & - & 4x_2 & + & 3x_3 & = & 15 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

Note

$$\begin{array}{lrcrcrcr} & -2x_1 & - & 4x_2 & + & 2x_3 & = & 8 \\ & 2x_1 & & & + & x_3 & = & 7 \\ \hline \text{(add)} & 0x_1 & - & 4x_2 & + & 3x_3 & = & 15 \end{array}$$

Example

Use some sequence of these three operations to solve the following system by eliminating variable(s). Keep track of the augmented matrix at each step.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \qquad \begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

We can use x_1 in the first equation to **eliminate** x_1 from equations E_2 and E_3. Perform $-2E_1 + E_2 \to E_2$ and then $-E_1 + E_3 \to E_3$.

$-2E_1 + E_2 \to E_2$

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & -4x_2 & + & 3x_3 & = & 15 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \qquad \begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -4 & 3 & 15 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

$-E_1 + E_3 \to E_3$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ -4x_2 + 3x_3 &= 15 \\ -x_2 + 2x_3 &= 10 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -4 & 3 & 15 \\ 0 & -1 & 2 & 10 \end{bmatrix}$$

$E_2 \leftrightarrow E_3$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ -x_2 + 2x_3 &= 10 \\ -4x_2 + 3x_3 &= 15 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -1 & 2 & 10 \\ 0 & -4 & 3 & 15 \end{bmatrix}$$

$-4E_2 + E_3 \to E_3$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ -x_2 + 2x_3 &= 10. \end{aligned}$$

$$-5x_3 = -25$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -1 & 2 & 10 \\ 0 & 0 & -5 & -25 \end{bmatrix}$$

$$-\tfrac{1}{5}E_3 \to E_3$$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ -x_2 + 2x_3 &= 10 \\ x_3 &= 5 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -1 & 2 & 10 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$-2E_3 + E_2 \to E_2$$

$$E_3 + E_1 \to E_1$$

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ -x_2 &= 0 \\ x_3 &= 5 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$-E_2 \to E_2$

$$\begin{aligned} x_1 + 2x_2 \quad &= 1 \\ x_2 \quad &= 0 \\ x_3 &= 5 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$-2E_2 + E_1 \to E_1$

$$\begin{aligned} x_1 \quad &= 1 \\ x_2 \quad &= 0 \\ x_3 &= 5 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

From this equivalent system, we see that the system has exactly one solution, $(1, 0, 5)$.

Elementary Row Operations

Elementary row operations are operations we can perform on the rows of a matrix. There are three of them, and they are analogous to operations on the equations in a system. We use a similar notation using R_i for the i^{th} row.

Elementary Row Operations

i Interchange row i and row j (**swap**), $R_i \leftrightarrow R_j$.

ii Multiply row i by any nonzero constant k (**scale**), $kR_i \to R_i$.

iii Replace row j with the sum of itself and k times row i (**replace**), $kR_i + R_j \to R_j$.

Row Equivalent Matrices

Definition

Two matrices are called **row equivalent** if one can be obtained from the other by performing a sequence of elementary row operations.

Theorem

If the augmented matrices of two linear systems of equations are row equivalent, then the linear systems of equations are equivalent (i.e., they have the same solution set).

A key here is *structure*!

Consider the following augmented matrix. Determine if the associated system is consistent or inconsistent. If it is consistent, determine the solution set.

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$

The system is

$$1x_1 + 0x_2 + 0x_3 = 3$$
$$0x_1 + 1x_2 + 0x_3 = 1$$
$$0x_1 + 0x_2 + 1x_3 = -2$$

The system is consistent, and the solution is $(3, 1, -2)$.

(b) $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ $\longrightarrow$

$$\underbrace{\begin{aligned} 1x_1 + 2x_2 + 0x_3 &= 3 \\ 0x_1 + 1x_2 - 1x_3 &= 4 \\ 0x_1 + 0x_2 + 0x_3 &= 3 \end{aligned}}_{0=3 \text{ is always false}}$$

The system is inconsistent.

At least one equation can't be satisfied.

(c) $\begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$1x_1 + 0x_2 - 2x_3 = -3$$
$$0x_1 + 1x_2 + 1x_3 = 4$$
$$\underbrace{0x_1 + 0x_2 + 0x_3 = 0}_{0=0 \text{ is always true}}$$

The system is consistent.

The solutions satisfy

$$x_1 = -3 + 2x_3$$
$$x_2 = 4 - x_3$$

and x_3 is any real number

$$\{(x_1, x_2, x_3) \mid x_1 = -3 + 2x_3 \text{ and } x_2 = 4 - x_3\}$$