# January 22 Math 3260 sec. 51 Spring 2024

## Section 1.3: Vector Equations

We defined vectors, specifically vectors in $\mathbb{R}^2$, and some basic arithmetic.

For vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $\mathbb{R}^2$ and scalar $c$ :

$$\mathbf{u} = \mathbf{v} \quad \text{if and only if} \quad u_1 = v_1 \quad \text{and} \quad u_2 = v_2.$$

$$c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}.$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

# Geometry of Algebra with Vectors

**Scalar Multiplication:** stretches or compresses a vector but can only change direction by an angle of 0 (if $c > 0$) or $\pi$ (if $c < 0$). We'll see that $0\mathbf{u} = (0, 0)$ for any vector $\mathbf{u}$.

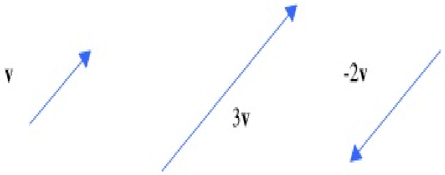



Figure: Scaled vectors are parallel. For nonzero vector $\mathbf{v}$, $c\mathbf{v}$ is stretched or compressed by a factor $|c|$ and flips 180° if $c$ is negative.

# Geometry of Algebra with Vectors

**Vector Addition:** The sum $\mathbf{u}+\mathbf{v}$ of two nonparallel vectors (each different from $(0,0)$) is the the fourth vertex of a parallelogram whose other three vertices are $(u_1, u_2)$, $(v_1, v_2)$, and $(0,0)$.

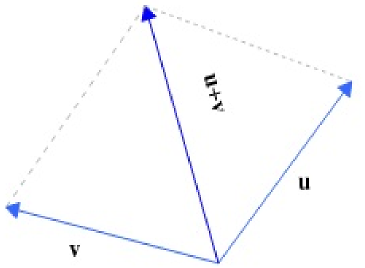



Figure: If $\mathbf{u}$ and $\mathbf{v}$ are nonzero and not parallel, they determine a paralelogram. The sum $\mathbf{u}+\mathbf{v}$ is a diagonal. (Note, the difference $\mathbf{u}-\mathbf{v}$ is the other diagonal.)

# Vectors in $\mathbb{R}^3$ (R three)

A vector in $\mathbb{R}^3$ is a $3 \times 1$ column matrix. For example

$$\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad \text{or} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Similar to vectors in $\mathbb{R}^2$, vectors in $\mathbb{R}^3$ are ordered triples.

$$\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = (1, 3, -1).$$

# Vectors in $\mathbb{R}^n$ (R n)

A vector in $\mathbb{R}^n$ for $n \geq 2$ is a $n \times 1$ column matrix. These are ordered $n$-tuples. For example

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

**The Zero Vector:** is the vector whose entries are all zeros. It will be denoted by $\mathbf{0}$ or $\vec{0}$ and is not to be confused with the scalar 0.

# Equivalence & Operations

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ be in $\mathbb{R}^n$ and $c$ is a scalar.

$$\text{Equivalence:} \quad \mathbf{u} = \mathbf{v} \quad \Leftrightarrow \quad u_i = v_i \quad \text{for each} \quad i = 1, \ldots, n$$

$$\text{Scalar Multiplication:} \quad c\mathbf{u} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix}$$

$$\text{Vector Addition:} \quad \mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

# Algebraic Properties on $\mathbb{R}^n$

## Algebraic Properties on $\mathbb{R}^n$

For every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^n$ and scalars $c$ and $d$

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv)[a] $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = d(c\mathbf{u}) = (cd)\mathbf{u}$

(viii) $1\mathbf{u} = \mathbf{u}$

[a] The term $-\mathbf{u}$ denotes $(-1)\mathbf{u}$.

# Linear Combination

**Definition**

A **linear combination** of vectors $\mathbf{v}_1, \ldots \mathbf{v}_p$ in $\mathbb{R}^n$ is a vector $\mathbf{y}$ of the form

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

where the scalars $c_1, \ldots, c_p$ are often called weights.

For example, suppose we have two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Some linear combinations include

$$3\mathbf{v}_1, \quad -2\mathbf{v}_1 + 4\mathbf{v}_2, \quad \frac{1}{3}\mathbf{v}_2 + \sqrt{2}\mathbf{v}_1, \quad \text{and} \quad \mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2.$$

## Example

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$. Determine if $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

The question can be restated as:

Do there exist scalars $c_1$ and $c_2$ such that $\vec{b} = c_1 \vec{a}_1 + c_2 \vec{a}_2$? Set it up.

$$c_1 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ -2c_1 \\ -c_1 \end{bmatrix} + \begin{bmatrix} 3c_2 \\ 0 \\ 2c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + 3c_2 \\ -2c_1 \\ -c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$$

$$\begin{aligned} c_1 + 3c_2 &= -2 \\ -2c_1 \phantom{+ 3c_2} &= -2 \\ -c_1 + 2c_2 &= -3 \end{aligned}$$

This is a linear system of equations.

We can use an augmented matrix

$$\left[\begin{array}{cc|c} 1 & 3 & -2 \\ -2 & 0 & -2 \\ -1 & 2 & -3 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

not a pivot column

From the rref we see that the system is consistent. Hence $\vec{b}$ is a linear combination of $\vec{a}_1$ and $\vec{a}_2$.

Moreover, $\vec{b} = \vec{a}_1 - \vec{a}_2$.

# Some Convenient Notation

Letting $\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$, and in general $\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$, for $j = 1, ..., n$, we can denote the $m \times n$ matrix whose columns are these vectors by

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Note that each vector $\mathbf{a}_j$ is a vector in $\mathbb{R}^m$.

# Vector and Matrix Equations

**Vector & Matrix Equations**

The vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}. \tag{1}$$

In particular, $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if the linear system whose augmented matrix is given in (1) is consistent.

# Span

**Definition**

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set of vectors in $\mathbb{R}^n$. The set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by

$$\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = \mathrm{Span}(S).$$

It is called the **subset of $\mathbb{R}^n$ spanned by (a.k.a. generated by)** the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

**Remark:** To say that a vector $\mathbf{b}$ is in $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ means that there exists a set of scalars $c_1, \ldots, c_p$ such that

$$\mathbf{b} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p.$$

# Equivalent Statements

Suppose $\mathbf{b}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are vectors in $\mathbb{R}^m$. The following are equivalent:

- $\mathbf{b}$ is in $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$,
- $\mathbf{b} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$ for some scalars $c_1, \ldots, c_p$,
- the vector equation $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$ has a solution,
- the linear system of equations whose augmented matrix is $[\mathbf{v}_1 \ \cdots \ \mathbf{v}_p \ \mathbf{b}]$ is consistent.

# Examples

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, and $\mathbf{a}_2 = \begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix}$.

(a) Determine if $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$.

we can use an augmented matrix

$[\vec{a}_1 \ \vec{a}_2 \ \vec{b}]$ to answer this question.

$$[\vec{a}_1 \ \vec{a}_2 \ \vec{b}] = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & 2 \\ 2 & -2 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

pivot column

The system having augmented matrix .

$[\vec{a}_1 \ \vec{a}_2 \ \vec{b}]$ is inconsistent.

Hence $\vec{b}$ is <u>not</u> in

span $\{\vec{a}_1, \vec{a}_2\}$.

(b) For what values of $k$, if any, is $\mathbf{b} = \begin{bmatrix} 5 \\ -5 \\ k \end{bmatrix}$ in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$?

We can set up the matrix $[\vec{a}_1 \ \vec{a}_2 \ \vec{b}]$.

$$[\vec{a}_1 \ \vec{a}_2 \ \vec{b}] = \begin{bmatrix} 1 & -1 & 5 \\ 1 & 4 & -5 \\ 2 & -2 & k \end{bmatrix} \qquad \begin{matrix} -R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 5 \\ 0 & 5 & -10 \\ 0 & 0 & k-10 \end{bmatrix}$$

Consistency requires the third column to <u>not</u> be a pivot column.

The system is only consistent if $k-10=0$ — i.e., if $k=10$.

$$\vec{b} = \begin{bmatrix} 5 \\ -5 \\ k \end{bmatrix} \text{ is in Span}\{\vec{a}_1, \vec{a}_2\}$$

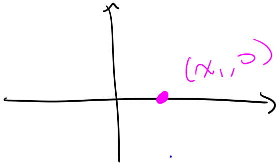
if $k=10$.

# Another Example

Give a geometric description of the subset of $\mathbb{R}^2$ given by $\text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$.

A vector in this set has the form $\vec{x} = x_1\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}x_1\\0\end{bmatrix}$.

This is a line. It's the x-axis.



# Span$\{\mathbf{u}\}$ in $\mathbb{R}^3$

If $\mathbf{u}$ is any nonzero vector in $\mathbb{R}^3$, then Span$\{\mathbf{u}\}$ is a line through the origin parallel to $\mathbf{u}$.

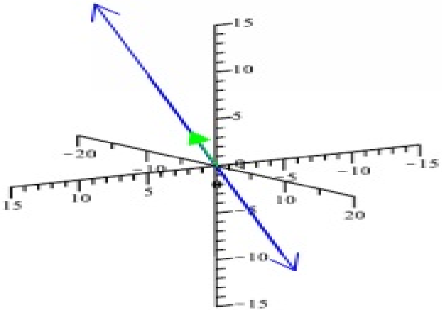
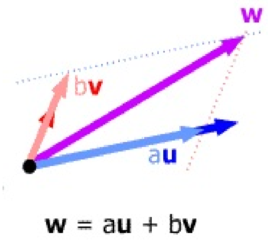



Figure: A nonzero vector $\mathbf{u}$ and the line Span$\{\mathbf{u}\}$ in $\mathbb{R}^3$.

# Span$\{\mathbf{u}, \mathbf{v}\}$ in $\mathbb{R}^3$

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero, and nonparallel vectors in $\mathbb{R}^3$, then Span$\{\mathbf{u}, \mathbf{v}\}$ is a plane containing the origin parallel to both vectors.



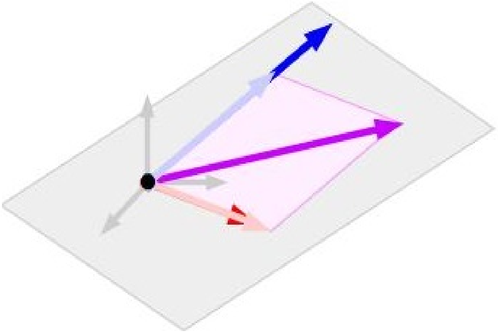



Figure: A vector $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$. If we let $a$ and $b$ vary, the collection of vectors Span$\{\mathbf{u}, \mathbf{v}\}$ is a plane.

# Example

Let $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (0, 2)$ in $\mathbb{R}^2$. Show that for every pair of real numbers $a$ and $b$, that $(a, b)$ is in Span$\{\mathbf{u}, \mathbf{v}\}$.

we can try to show the system w/ augmented matrix $[\vec{u}\ \vec{v}\ \vec{x}]$ is always consistent when $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$.

$$[\vec{u}\ \vec{v}\ \vec{x}] = \begin{bmatrix} 1 & 0 & a \\ 1 & 2 & b \end{bmatrix} \quad -R_1 + R_2 \to R_2$$

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 2 & b-a \end{bmatrix} \quad \tfrac{1}{2}R_2 \to R_2$$

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & \frac{b-a}{2} \end{bmatrix}$$ This is an rref.

The last column will never be a pivot column. Hence the system is always consistent.

That is $\begin{bmatrix} a \\ b \end{bmatrix}$ is in Span $\{\vec{u}, \vec{v}\}$. Span $\{\vec{u}, \vec{v}\}$ is $\mathbb{R}^2$.