January 31 Math 3260 sec. 51 Spring 2024

Section 1.7: Linear Independence

Definition:Linear Independence

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

Remark: Alternatively, we can say that the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is **linearly dependent** if there exists a set of weights $c_1, c_2, \ldots, c_p$, *at least one of which is nonzero*, such that the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}.$$

is satisfied.

Theorem on Linear Independence

Theorem:

The columns of a matrix A are linearly **independent** if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Remark: We can use this result as a tool. Given any set of vectors in $\mathbb{R}^n$, we can always create a matrix from them by just using them as columns.

Example

(c) Determine if the set of vectors is linearly dependent or linearly independent. If dependent, find a linear dependence relation.

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$$

We called the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and created a matrix A having them as its columns. We also did row reduction and found

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & 3 & 2 & 0 \\ 0 & 1 & 3 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = \frac{-1}{3} x_4$

$x_2 = -2 x_4$

$x_3 = \frac{2}{3} x_4$

x_4 - free

Example Continued...

We see that the system $A\mathbf{x} = \mathbf{0}$ would have nontrivial solutions because there's a non-pivot column among the first four columns. So the set of vectors is **linearly dependent.** Let's see how we can use the rref to get a **linear dependence relation.**

From the rref, we can write a vector equation

$$-\frac{1}{3}x_4\vec{v}_1 - 2x_4\vec{v}_2 + \frac{2}{3}x_4\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$$

Pick any value for x_4 to get a linear dependence relation. For example, taking $x_4 = -3$

we get

$$\vec{v}_1 + 6\vec{v}_2 - 2\vec{v}_3 - 3\vec{v}_4 = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_4 = \frac{1}{3}\vec{v}_1 + 2\vec{v}_2 - \frac{2}{3}\vec{v}_3$$

Observation

In a previous example we used the parametric form of a solution to a system to write the parametric vector form.

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & & & + & x_4 & = & 2 \\ 3x_1 & - & 6x_2 & + & x_3 & - & x_4 & = & 7 \end{array}$$

has solutions given by

$$\begin{array}{rcl} x_1 & = & 2+2x_2-x_4 \\ x_2 & - & \text{free} \\ x_3 & = & 1+4x_4 \\ x_4 & - & \text{free} \end{array} \implies \mathbf{x} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{p}} \underbrace{+ s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 4 \\ 1 \end{bmatrix}}_{\mathbf{v}_h}, \quad s,t \in \mathbb{R}.$$

Remark: Note that decomposing the part of the solution to the homogeneous equation by isolting the two free variables guarantees that the vectors in $\mathbf{v}_h$ will be linearly independent. Each one will have a 1 in the entry corresponding to one free variable and zero(s) in the entry(ies) corresponding to all other free variable(s).

Theorem

Theorem

An indexed set of two or more vectors is linearly dependent if and only if at least one vector in the set is a linear combination of the others in the set.

Example: Let **u** and **v** be any nonzero vectors in $\mathbb{R}^3$. Show that if **w** is any vector in Span$\{\mathbf{u}, \mathbf{v}\}$, then the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly **dependent.**

Since $\vec{w}$ is in Span$\{\vec{u}, \vec{v}\}$, there exist scalars c_1, c_2 such that

$$\vec{w} = c_1\vec{u} + c_2\vec{V}.$$

we can get a linear dependence relation

from this.

$$c_1\vec{u} + c_2\vec{v} - \vec{w} = \vec{0}$$

The coefficient of $\vec{w}$ is $-1 \neq 0$.

This is a lin. dep. relation so

$\{\vec{u}, \vec{v}, \vec{w}\}$ is lin. dependant.

Caveat!

A set may be linearly dependent even if all proper subsets are linearly independent. For example, consider

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Each set $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. (You can easily verify this.)

However,

$$\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1 \quad \text{i.e.} \quad \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0},$$

so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

This means that you can't just consider two vectors at a time.

Two More Theorems

Theorem:

If a set contains more vectors than there are entries in each vector, then the set is linearly **dependent**. That is, if $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is a set of vector in $\mathbb{R}^n$, and $p > n$, then the set is linearly dependent.

For example, if you have 7 vectors, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7\}$, and each of these is a vector in $\mathbb{R}^5$, i.e., $\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \\ v_{41} \\ v_{51} \end{bmatrix}$ and so forth, then they must be **linearly dependent** because $7 > 5$.

Two More Theorems

Theorem:

Any set of vectors that contains the zero vector is linearly **dependent.**

Consider the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p, \mathbf{0}\}$ in $\mathbb{R}^n$. Note that

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p + 1\mathbf{0} = \mathbf{0}$$

is a **linear dependence relation** because the last coefficient $c_{p+1} = 1$ is nonzero. It doesn't matter what the other vectors are or what the values of p and n are relative to one another!

Examples

Without doing any computations, determine, with justification, whether the given set is linearly dependent or linearly independent.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$

This is 4 vectors in $\mathbb{R}^3$.

$4 > 3$, they are <u>lin.</u>

<u>dependent</u>.

Examples

(b) $$\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -8 \\ 1 \end{bmatrix}, \right\}$$

This set contains $\vec{0}$, it is

lin. dependent