January 8 Math 3260 sec. 51 Spring 2024

A Random Motivational Example

Plop plop, fizz fizz, oh what a relief it is[1].

$$\underbrace{\text{NaHCO}_3}_{\text{sodium bicarbonate}} + \underbrace{\text{H}_3\text{C}_6\text{H}_5\text{O}_7}_{\text{citric acid}} \longrightarrow \underbrace{\text{Na}_3\text{C}_6\text{H}_5\text{O}_7}_{\text{sodium citrate}} + \underbrace{\text{H}_2\text{O}}_{\text{water}} + \underbrace{\text{CO}_2}_{\text{carbon dioxide}}.$$

This is an unbalanced chemical equation that describes effervescence of a commercial antacid medication.

Question: How many molecules of each substance result in a balanced equation?

[1] Sodium bicarbonate and citric acid dissolved in water produces sodium citrate, water, and carbon dioxide.

Motivating Example: Balancing Atoms

$$\underline{x_1}\,\mathrm{NaHCO_3} + \underline{x_2}\,\mathrm{H_3C_6H_5O_7} \longrightarrow \underline{x_3}\,\mathrm{Na_3C_6H_5O_7} + \underline{x_4}\,\mathrm{H_2O} + \underline{x_5}\,\mathrm{CO_2}$$

We can introduce a 4-tuple $\begin{bmatrix} \text{Na} \\ \text{H} \\ \text{C} \\ \text{O} \end{bmatrix}$ and create an equation for the unknowns x_1, x_2, x_3, x_4, and x_5.

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 8 \\ 6 \\ 7 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 5 \\ 6 \\ 7 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

This is an example of the types of equations we want to consider.

We'll work in a variety of settings...

Linear sys.

$$\begin{array}{rcrcrcrcrcl} x_1 & & & - & 3x_3 & & & & & = & 0 \\ x_1 & + & 8x_2 & - & 5x_3 & - & 2x_4 & & & = & 0 \\ x_1 & + & 6x_2 & - & 6x_3 & & & - & x_5 & = & 0 \\ 3x_1 & + & 7x_2 & - & 7x_3 & - & x_4 & - & 2x_5 & = & 0 \end{array}$$

Matrix eqns.

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 1 & 8 & -5 & -2 & 0 \\ 1 & 6 & -6 & 0 & -1 \\ 3 & 7 & -7 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

More Matrices

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 1 & 8 & -5 & -2 & 0 & 0 \\ 1 & 6 & -6 & 0 & -1 & 0 \\ 3 & 7 & -7 & -1 & -2 & 0 \end{bmatrix}$$

Vector eqns.

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 8 \\ 6 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -5 \\ -6 \\ -7 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ 0 \\ -1 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Questions:

- Is there a set of numbers $x_1, \ldots, x_5$ that satisfy all of the equations?
- If there is a set of numbers, is it the only one?
- Are there simple algorithms we can use to answer these questions?

These are some of the questions addressed by Linear Algebra. We'll also consider two main abstractions:

Vector Spaces and **Linear Transformations**.

Section 1.1: Systems of Linear Equations

We begin with a linear (*algebraic*) equation in *n real* variables x_1, x_2, ..., x_n for some positive integer n.

Definition

A **linear equation** in the variables $x_1, \dots, x_n$ is one that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, \dots, a_n$ are real (or complex) constants called the *coefficients*, and b is a constant.

In general, the coefficients and the right hand side b are known.

Linear Equation in n Variables

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

Notice the main structure on the left side. The unknowns/variables $(x_1, \ldots, x_n)$ are

- multipled by numbers (a.k.a. coefficients), and
- added together.

Other types of actions (squaring, multiplying variables, taking variable's reciprocal, etc.) aren't allowed if an equation is **linear**.

Examples of Equations that are or are not Linear

$$2x_1 = 4x_2 - 3x_3 + 5 \quad \text{and} \quad 12 - \sqrt{3}(x + y) = 0$$

These equations are linear.

$2x_1 - 4x_2 + 3x_3 = 5$

$\sqrt{3}x + \sqrt{3}y = 12$

Note that both can be written in the format from the definition. The only operations on the variables are (1) multiply by constants and (2) add.

Examples of Equations that are or are not Linear

$$x_1 + 3x_3 = \frac{1}{x_2} \quad \text{and} \quad xyz = \sqrt{w}$$

These equations are NOT linear.

Reciprocals are not linear

The product xyz is nonlinear

as is $\sqrt{w}$.

Definition

A **linear system** (or linear system of equations) is a collection of linear equations in the same variables.

The equations in a linear system are considered together as one *object*.

Example 1:
$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & - & 3x_3 & + & x_4 & = & -3 \\ -x_1 & + & 3x_2 & + & 4x_3 & - & 2x_4 & = & 8 \end{array}$$

Example 1 is a linear system that has two equations in four variables.

Example 2:
$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4 \\ 3x & & & + & 12z & = & 0 \\ 2x & + & 2y & - & 5z & = & -6 \end{array}$$

Example 2 is a linear system that has three equations in three variables.

In this course, we'll mostly use a single variable name with subscripts, i.e., x_1, x_2, x_3 as opposed to x, y, z.

Some Preliminary Terms

Consider the system of m equations in the variables $x_1, \ldots, x_n$

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & \vdots & & \vdots & \cdots & \vdots & & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array} \tag{1}$$

Definitions: Solution and Solution Set

A **solution** of (1) is an ordered list of numbers $(s_1, s_2, \ldots, s_n)$ that reduce each equation in the system to a true statement upon substitution[a].

The **solutions set** of (1) is the set of all possible solutions.

[a] It is assumed that substitution means setting $x_1 = s_1$, $x_2 = s_2$ and so forth.

Some Preliminary Terms

Consider the system of m equations in the variables $x_1, \dots, x_n$

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
 & \vdots & & \vdots & \cdots & \vdots & & = & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m.
\end{array}$$

Definition: Equivalent Systems

Two linear systems are called **equivalent** (or equivalent systems) if they have the same solution set.

Remark: We'll often use some process to rewrite a system in terms of an equivalent system for which the solution(s) is more obvious.

An Example

$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & -1 \\ -4x_1 & + & 2x_2 & = & 2 \end{array}$$

(a) Show that $(1, 3)$ is a solution.

Set $x_1 = 1$ and $x_2 = 3$

$2(1) - (3) \stackrel{?}{=} -1$

$2 - 3 = -1$ it's true

$-4(1) + 2(3) \stackrel{?}{=} 2$

$-4 + 6 = 2$ it's also true.

An Example Continued

$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & -1 \\ -4x_1 & + & 2x_2 & = & 2 \end{array}$$

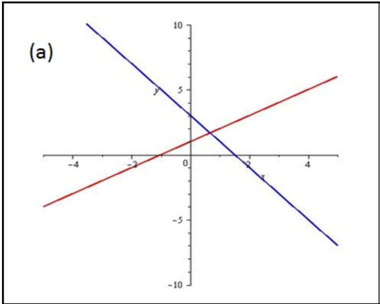
(b) The **solution set** for this system is

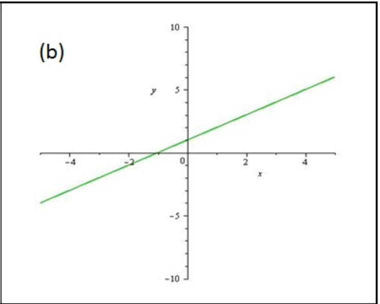
$$\left\{ (x_1, x_2) \,|\, x_1 = -\frac{1}{2} + \frac{1}{2}x_2 \right\}.$$

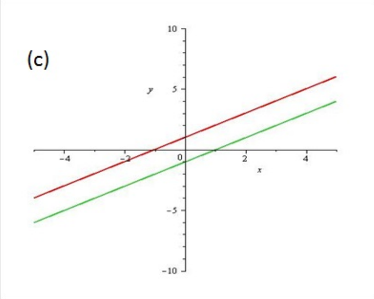
Notice that setting $x_1 = -\frac{1}{2} + \frac{1}{2}x_2$ in each equation we get the pair of true statements

$$\begin{array}{rcrcr} 2\left(-\frac{1}{2} + \frac{1}{2}x_2\right) & - & x_2 & = & -1, \qquad \text{and} \\ -4\left(-\frac{1}{2} + \frac{1}{2}x_2\right) & + & 2x_2 & = & 2. \end{array}$$

The Geometry of 2 Equations with 2 Variables







Graphical Illustration of Solution Cases

(a) $\begin{aligned} x - y &= -1 \\ 2x + y &= 3 \end{aligned}$ One solution case. Intersecting lines

(b) $\begin{aligned} x - y &= -1 \\ 2x - 2y &= -2 \end{aligned}$ Infinitely many solutions. One line

(c) $\begin{aligned} x - y &= -1 \\ 2x - 2y &= 2 \end{aligned}$ No solutions case. Parallel lines

Theorem

Theorem

For a linear system, exactly one of the following holds. The system has

i no solution, or

ii exactly one solution, or

iii infinitely many solutions.

A system is called **inconsistent** if it does not have any solutions (case i), and it's called **consistent** if it has any solution(s) (cases ii & iii).

Note: This theorem speaks to those two big questions:

- Existence: Is there a solution/does a solution exist?
- Uniqueness: Is there a unique solution or multiple solutions?