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Section 4.2: Null & Column Spaces, Row Space, Linear Transformations

In this section, we'll consider some subspaces (of $\mathbb{R}^n$ or $\mathbb{R}^m$) associated with a matrix, and extend the notion of a linear transformation to functions between arbitrary vector spaces.

Definition

Definition: Let A be an $m \times n$ matrix. The **null space** of A, denoted by Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. That is

$$\text{Nul}\, A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

Example

Express Nul(A) in terms of a spanning set, where $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 7 \end{bmatrix}$.

(**Note:** a spanning set for Nul(A) is called an *explicit description* of it.)

Nul(A) is the set of all vectors $\vec{x}$ in $\mathbb{R}^3$ such that $A\vec{x} = \vec{0}$. We can use an augmented matrix $[A \ \vec{0}]$.

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 1 & 2 & 7 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}.$$

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then

$x_1 = -3x_3$

$x_2 = -2x_3$

x_3 is free

$$\vec{x} = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}.$$

So $\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}.$

The Null Space as a Subspace

Theorem:

If A is an $m \times n$ matrix, then $\text{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

$\vec{x}$ must be in $\mathbb{R}^n$ for $A\vec{x}$ to be defined.

Nul(A) is a span, hence a subspace.

or refer to worksheet 11.

Column Space

Definition:

The **column space** of an $m \times n$ matrix A, denoted $\operatorname{Col} A$, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$, then

$$\operatorname{Col} A = \operatorname{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}.$$

Theorem:

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Corollary

$\operatorname{Col} A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.

Example

Find a matrix A such that $W = \text{Col}\, A$ where

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}.$$

We can characterize an element of W as a linear combo of fixed vectors and then as a product $A\vec{x}$. Consider any vector from W.

$$\begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} = \begin{bmatrix} 6a \\ a \\ -7a \end{bmatrix} + \begin{bmatrix} -b \\ b \\ 0 \end{bmatrix}$$

$$= a\begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

Letting $A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$,

$W = \mathrm{Col}(A)$.

Row Space

Definition:

The **row space**, denoted $\text{Row}\, A$, of an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

Theorem

If two matrices A and B are row equivalent, then their row spaces are the same.

Example

Find two spanning sets for $\mathrm{Row}(A)$ given

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By the last theorem, both matrices have the same row space (since A is row equivalent to its rref).

$$\mathrm{Row}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 2 \\ 4 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -8 \\ 6 \end{bmatrix} \right\} = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Comparing Col(A) and Nul(A)

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

(a) If Col A is a subspace of $\mathbb{R}^k$, what is k?

$\mathrm{Col}(A) = \mathrm{Span}\{\text{columns of } A\}$. $k = 3$

(b) If Nul A is a subspace of $\mathbb{R}^k$, what is k?

we need $A\vec{x}$ to be defined. $k = 4$.

Example Continued...

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

(c) Is **u** in Nul A? Could **u** be in Col A?

$\vec{u}$ is in Nul(A) if $A\vec{u} = \vec{0}$.

$A\vec{u} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \vec{0} \implies \vec{u}$ is not in Nul(A).

$\vec{u}$ is in $\mathbb{R}^4$ whereas Col(A) is a subspace of $\mathbb{R}^3$. $\vec{u}$ can't be in Col(A).

Example Continued...

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

(d) Is **v** in Col A? Could **v** be in Nul A?

$A\vec{v}$ isn't defined, so $\vec{v}$ can't be in Nul(A).

$\vec{v}$ is in Col(A) if $A\vec{x} = \vec{v}$ is consistent.

We can use an augmented matrix,

$$[A \ \vec{v}] = \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \xrightarrow{\text{rref}}$$

$$\begin{bmatrix} 1 & 0 & 9 & 0 & 5 \\ 0 & 1 & -5 & 0 & -30/17 \\ 0 & 0 & 0 & 1 & 1/17 \end{bmatrix}$$

not a pivot column

$A\vec{x} = \vec{v}$ is consistent. Hence

$\vec{v}$ is in $\mathrm{Col}(A)$.

Fundamental Subspaces

People often refer to four fundamental subspaces associated with an $m \times n$ matrix. The fourth one is the null space of A^T.

Remark: Since the rows of A are the columns of A^T and vice versa, it's not surprising that

$$\mathrm{Col}(A) = \mathrm{Row}(A^T) \quad \text{and} \quad \mathrm{Row}(A) = \mathrm{Col}(A^T).$$

Remark: We can summarize that for $m \times n$ matrix A

$$\mathrm{Col}(A) \quad \text{and} \quad \mathrm{Nul}(A^T) \text{ are subspaces of } \mathbb{R}^m,$$

and

$$\mathrm{Row}(A) \quad \text{and} \quad \mathrm{Nul}(A) \text{ are subspaces of } \mathbb{R}^n.$$

Linear Transformation

Definition:

Let V and W be vector spaces. A linear transformation $T: V \to W$ is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W such that

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v}$ in V, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every $\mathbf{u}$ in V and scalar c.

Remark: The only difference between this definition and our previous one is that the domain and codomain spaces can be any vector spaces.

Example

Let $C^1(\mathbb{R})$ denote[1] the set of all real valued functions that are differentiable and $C^0(\mathbb{R})$ the set of all continuous real valued functions. Note that differentiation is a linear transformation. That is

$$D : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R}), \quad D(f) = f'$$

satisfies the two conditions in the previous definition.

We know from calculus that if f and g are differentiable and c is a scalar, then

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x) \quad \text{and} \quad \frac{d}{dx}(cf(x)) = cf'(x).$$

Using the current notation, we can write these statements like

$$D(f + g) = D(f) + D(g) \quad \text{and} \quad D(cf) = cD(f).$$

[1]This could also be written as $C^1(-\infty, \infty)$.

Example

Consider the derivative transformation on $C^1(\mathbb{R})$

$$D : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$$

$$f \mapsto f'$$

Characterize the subset of $C^1(\mathbb{R})$ such that $D(f) = 0$.

This is the set of constant functions on $\mathbb{R}$. $f(x) = C$ for all x where C is some constant.

Note: The zero vector in $C^0(\mathbb{R})$ is the function $f_0(x) = 0$ for all real x.

Range and Kernel

Definition:

The **range** of a linear transformation $T : V \longrightarrow W$ is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V. (The set of all images of elements of V.)

A column space is a **range**.

Definition:

The **kernel** of a linear transformation $T : V \longrightarrow W$ is the set of all vectors $\mathbf{x}$ in V such that $T(\mathbf{x}) = \mathbf{0}$. (The analog of the null space of a matrix.)

A null space is a **kernel**.

Range & Kernel as Subspaces

Theorem:

Given a linear transformation $T : V \longrightarrow W$,

- the range of T is a subspace of W,
- and the kernel of T is a subspace of V.

Remark: This generalizes the result for column and null spaces. If $T : \mathbb{R}^n \to \mathbb{R}^m$, $T(\mathbf{x}) = A\mathbf{x}$. Then $\text{Col}(A)$ is the range of T and is a subspace of $\mathbb{R}^m$. And $\text{Nul}(A)$ is the kernel of T and is a subspace of $\mathbb{R}^n$.

Example

Consider $T : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$ defined by

$$T(f) = \frac{df}{dx} + \alpha f(x), \quad \alpha \text{ a fixed constant.}$$

(a) Express the equation that a function y must satisfy if y is in the kernel of T.

y is in the kernel if $T(y) = 0$.

$$T(y) = \frac{dy}{dx} + \alpha y$$

y is in the kernel of T if

$$\frac{dy}{dx} + \alpha y = 0$$

$$T : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R}), \quad T(f) = \frac{df}{dx} + \alpha f(x)$$

(b) Show that for any scalar c, $y = ce^{-\alpha x}$ is in the kernel of T.

To be in the kernel, we need $\frac{dy}{dx} + \alpha y = 0$.

If $y = ce^{-\alpha x}$, then $\frac{dy}{dx} = ce^{-\alpha x}(-\alpha)$

$= -\alpha c e^{-\alpha x}$.

$$\frac{dy}{dx} + \alpha y = -\alpha c e^{-\alpha x} + \alpha\left(c e^{-\alpha x}\right) = 0$$

So $y = ce^{-\alpha x}$ is in the kernel of T.