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Section 4.3: Linearly Independent Sets and Bases

Definition:

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in a vector space V is said to be **linearly independent** if the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \qquad (1)$$

has only the trivial solutions $c_1 = c_2 = \cdots = c_p = 0$.

The set is **linearly dependent** if there exist a nontrivial solution (at least one of the weights c_i is nonzero).

If there is a nontrivial solution $c_1, \ldots, c_p$, then equation (1) is called a **linear dependence relation**.

Linearly Dependent Sets

Theorem:

Consider the ordered set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in a vector space V, where $p \geq 2$ and $\mathbf{v}_1 \neq \mathbf{0}$. This set is **linearly dependent** if and only if there is some $j > 1$ such that $\mathbf{v}_j$ is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

This says that

1. If one of the vectors, say $\mathbf{v}_j$ can be written as a linear combo of the ones that come before it, the set is linearly dependent, and
2. if the set is linearly dependent, it must be possible to write one of the vectors as a linear combo of the others.

Example

Determine if the set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly dependent or independent in $\mathbb{P}_2$, where

$$\mathbf{p}_1 = 1, \quad \mathbf{p}_2 = 2t, \quad \text{and} \quad \mathbf{p}_3 = t - 3.$$

Note that $\vec{p}_3 = \frac{1}{2}\vec{p}_2 - 3\vec{p}_1$.

Hence $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ is linearly dependent. In fact,

$$3\vec{p}_1 - \frac{1}{2}\vec{p}_2 + \vec{p}_3 = \vec{0}$$

is a linear dependence relation.

Basis

Definition:

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** of H provided

(i) $\mathcal{B}$ is linearly independent, and

(ii) $H =$Span$(\mathcal{B})$.

Remark: We can think of a basis as a *minimal spanning set*. All of the *information* needed to construct vectors in H is contained in the basis, and none of this information is repeated.

Prelude to a Spanning Set Theorem

Example: Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be vectors in a vector space V, and suppose that

(1) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and
(2) $\mathbf{v}_3 = \mathbf{v}_1 - 2\mathbf{v}_2$.

Show that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

We need to show that if $\vec{u}$ is any vector in H, then $\vec{u}$ is in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$.

Since $\vec{u}$ is in H,

$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ for some

scalars c_1, c_2, c_3.

Since $\vec{v}_3 = \vec{v}_1 - 2\vec{v}_2$, we can write

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(\vec{v}_1 - 2\vec{v}_2).$$

$$= (c_1 + c_3)\vec{v}_1 + (c_2 - 2c_3)\vec{v}_2.$$

$$= k_1\vec{v}_1 + k_2\vec{v}_2 \quad \text{where } k_1 = c_1 + c_3,\ k_2 = c_2 - 2c_3$$

So $\vec{u}$ is in $\mathrm{Span}\{\vec{v}_1, \vec{v}_2\}$.

Spanning Set Theorem

Theorem

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set in a vector space V and $H =$Span(S).

a. If one of the vectors in S, say $\mathbf{v}_k$ is a linear combination of the other vectors in S, then the subset of S obtained by eliminating $\mathbf{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, then some subset of S is a basis for H.

If we start with a spanning set, we can eliminate *duplication* to construct a **basis**.

Column Space

Find a basis for the column space matrix B that is in reduced row echelon form

$$B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Call the columns $\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \vec{b}_5$ in the order shown. Then

$\mathrm{Col}(B) = \mathrm{Span}\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \vec{b}_5\}$.

$\vec{b}_2 = 4\vec{b}_1$. We can remove $\vec{b}_2$.

$\{\vec{b}_1, \vec{b}_3\}$ is lin. independent.

$\vec{b}_4 = 2\vec{b}_1 - \vec{b}_3$. We can remove $\vec{b}_4$.

$\{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$ is lin. independent.

$\{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$ is a basis for

$\text{Col}(B)$.

This basis contains the pivot

columns of B.

Using the rref

Theorem:

If $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$ and $B = [\mathbf{b}_1 \cdots \mathbf{b}_n]$ are row equivalent matrices, then Nul $A =$ Nul B. That is, the equations

$$A\mathbf{x} = \mathbf{0} \quad \text{and} \quad B\mathbf{x} = \mathbf{0}$$

have the same solution set.

Remark: This means that $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ and $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ have **exactly the same linear dependence relationships**!

Remark: We've actually already used this when we used an rref to characterize a null space.

A Basis for a Column Space

Theorem

Let A be an $m \times n$ matrix. The pivot columns of a matrix A form a basis of $\mathrm{Col}(A)$.

Caveat: This means we can use row reduction to identify a basis, but the vectors in the basis will be from the original matrix A.

Example

Consider the matrix A shown with a row equivalent rref. Find a basis for $\mathrm{Col}(A)$.

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the rref, the pivot columns are 1, 3, and 5. A basis for Col(A) is

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}.$$

Basis for a Row Space

Theorem:

If two matrices A and B are row equivalent, then their row spaces are the same.

Remark: This tells us that a basis for the row space of an $m \times n$ matrix A is the set of nonzero rows of its rref.

Remark: Note how this is different from the column space. For $\mathrm{Col}(A)$, take the vectors from A, but for $\mathrm{Row}(A)$ take the vectors from the rref.

Find a basis for Row(A)

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can take the nonzero rows from the rref. A basis for Row(A) is

$$\left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Bases for $\mathrm{Col}(A)$, $\mathrm{Row}(A)$, and $\mathrm{Nul}(A)$

Given a matrix A, find the rref. Then

- The pivot columns of the original matrix A give a basis for $\mathrm{Col}(A)$.
- The nonzero rows of rref(A) give a basis for $\mathrm{Row}(A)$.
- Use the rref to solve $A\mathbf{x} = \mathbf{0}$ to identify a basis for $\mathrm{Nul}(A)$.

Standard or Elementary Basis in $\mathbb{R}^n$

The columns of the $n \times n$ identity matrix provide an obvious basis for $\mathbb{R}^n$. This is called the **standard basis** or the **elementary basis** for $\mathbb{R}^n$.

The examples in $\mathbb{R}^2$ and $\mathbb{R}^3$ are

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{respectively.}$$

When we want an ordered basis, we order these in the obvious way, $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$.

Other Vector Spaces

The set $\{1, t, t^2, t^3\}$ is a basis[1] for $\mathbb{P}_3$.

Notice that for any vector $\mathbf{p}$ in $\mathbb{P}_3$,

$$\mathbf{p}(t) = p_0 1 + p_1 t + p_2 t^2 + p_3 t^3.$$

This is a linear combination of 1, t, t^2, and t^3. We already know that the zero polynomial

$$\mathbf{0}(t) = 0 1 + 0t + 0t^2 + 0t^3.$$

That is, the equation

$$c_0 + c_1 t + c_2 t^2 + c_3 t^3 = 0 \quad \Leftrightarrow \quad c_0 = c_1 = c_2 = c_3 = 0$$

[1] The set $\{1, t, \ldots, t^n\}$ is called the **standard basis** for $\mathbb{P}_n$

Other Vector Spaces

The set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M_{2\times 2}$.

The exercise is left to the reader. It must be shown that

- every matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be written as a linear combination of these vectors and
- this is a linearly independent set.