March 25 Math 3260 sec. 51 Spring 2024

Section 4.4: Coordinate Systems

Theorem:

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis for a vector space V. Then for each vector $\mathbf{x}$ in V, there is a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n.$$

- **Remark:** It's clear that each vector can be written as a linear combination because a basis is a spanning set.
- **Remark:** This is saying that it can only be done in one way—that is, there is only one set of numbers $c_1, \ldots, c_n$.

Uniqueness of Coefficients

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis for a vector space V and let $\mathbf{x}$ be a vector in V. If

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots c_n\mathbf{b}_n \quad \text{and} \\ \mathbf{x} &= a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots a_n\mathbf{b}_n, \end{aligned}$$

show that $a_1 = c_1, a_2 = c_2, \ldots, a_n = c_n$.

We'll create a homogeneous equation by subtracting one line from the other.

$$\vec{0} = (c_1 - a_1)\vec{b}_1 + (c_2 - a_2)\vec{b}_2 + \cdots + (c_n - a_n)\vec{b}_n$$

Since $\mathcal{B}$ is linearly independent, all coefficients must be zero.

Hence $c_1 - a_1 = 0 \Rightarrow a_1 = c_1$

$$c_2 - a_2 = 0 \Rightarrow a_2 = c_2$$

$$\vdots$$

$$c_n - a_n = 0 \Rightarrow c_n = a_n$$

That is, the two expressions have exactly the same coefficients.

Consequence of Linear Independence

$$\text{Let} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It is true that $\mathbb{R}^2 = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Consider $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Note that we can write $\mathbf{x}$ in two different ways

$$\begin{aligned} \mathbf{x} &= 2\mathbf{v}_1 + 3\mathbf{v}_2 + 0\mathbf{v}_3 \quad \text{and} \\ \mathbf{x} &= 1\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3. \end{aligned}$$

Why doesn't this contradict our theorem?

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a spanning set, but it's not a basis because it's linearly dependent.

Definition: Coordinate Vectors

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an **ordered** basis of the vector space V. For each $\mathbf{x}$ in V we define the **coordinate vector of x relative to the basis** $\mathcal{B}$ to be the unique vector $(c_1, \ldots, c_n)$ in $\mathbb{R}^n$ whose entries are the weights $\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n$.

We'll use the notation $[\mathbf{x}]_\mathcal{B}$; that is $[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$.

Big Idea!

The vector $\mathbf{x}$ can be any sort of vector (from any sort of vector space), but

$$\boxed{[\mathbf{x}]_\mathcal{B} \quad \text{is a vector in} \quad \mathbb{R}^n}$$

Example

Consider the basis $\mathcal{B} = \{1, t, t^2, t^3\}$ (in that order) for $\mathbb{P}_3$.
Determine $[\mathbf{p}]_{\mathcal{B}}$ for

(a) $\mathbf{p}(t) = 3 - 4t^2 + 6t^3$

With 4 basis elements, the coordinate vector will be in $\mathbb{R}^4$. Note that

$$\vec{p}(t) = 3(1) + 0t + (-4)t^2 + 6t^3$$

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ -4 \\ 6 \end{bmatrix}.$$

$\mathcal{B} = \{1, t, t^2, t^3\}$

Determine $[\mathbf{p}]_{\mathcal{B}}$ for

(b) $\mathbf{p}(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

An Alternative Basis for $\mathbb{R}^2$

Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ where $\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2$

$$\begin{bmatrix} 4 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

we see that $\vec{x} = (\text{some matrix}) [\vec{x}]_{\mathcal{B}}$.

let's use a matrix inverse to solve this.

we need: $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 4+5 \\ -4+10 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\left[\begin{bmatrix} 4 \\ 5 \end{bmatrix} \right]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Change of Coordinates Matrix

Note in this example that

$$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

where $P_{\mathcal{B}}$ is the matrix having the basis vectors from $\mathcal{B}$ as its columns.

$$P_{\mathcal{B}} = [\mathbf{b}_1 \ \mathbf{b}_2]$$

Definition

Given an ordered basis $\mathcal{B}$ in $\mathbb{R}^n$, the matrix

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n].$$

is called the **change of coordinates matrix** for the basis $\mathcal{B}$ (or from the basis $\mathcal{B}$ to the standard basis).

Change of Coordinates in $\mathbb{R}^n$

Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis of $\mathbb{R}^n$. Then the change of coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the linear transformation defined by

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$$

where the matrix

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n].$$

Remark: By the Invertible Matrix Theorem, we know that a change of coordinates mapping is a **one to one** transformation of $\mathbb{R}^n$ **onto** $\mathbb{R}^n$.

Example

$\vec{x} = P_{\mathcal{B}} [\vec{x}]_{\mathcal{B}}$ and $[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \vec{x}$

For $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$, we have

$$P_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad P_{\mathcal{B}}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

(a) Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \vec{x} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(b) Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\vec{x} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(c) Find $\mathbf{x}$ if $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\vec{x} = P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$$

$$\vec{x} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

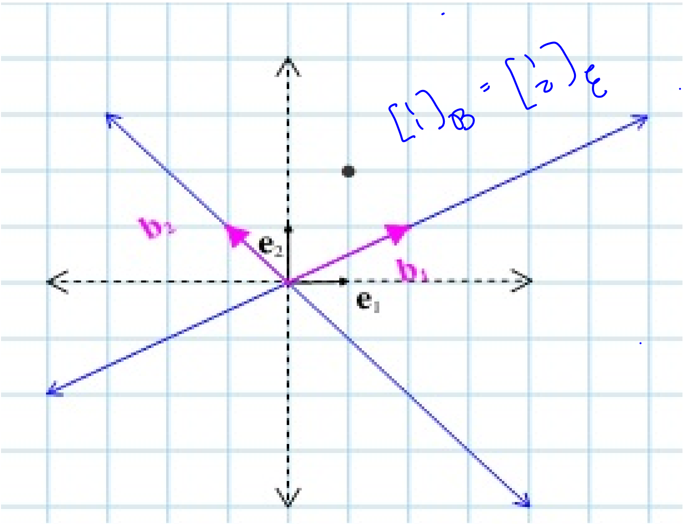

Figure: $\mathbb{R}^2$ shown using elementary basis $\{(1,0),(0,1)\}$ and with the alternative basis $\{(2,1),(-1,1)\}$.

We can make new graph paper using this basis.

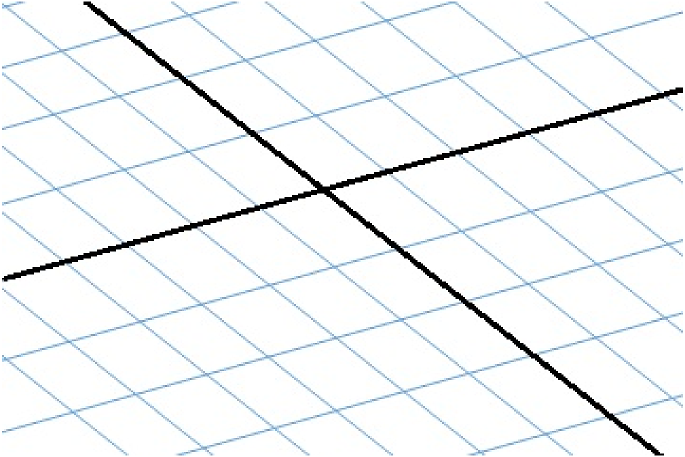

Figure: Graph paper constructed using the basis $\{(2,1),(-1,1)\}$.