March 27 Math 3260 sec. 51 Spring 2024

Section 4.4: Coordinate Systems

Definition: Coordinate Vectors

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an **ordered** basis of the vector space V. For each $\mathbf{x}$ in V we define the **coordinate vector of x relative to the basis** $\mathcal{B}$ to be the unique vector $(c_1, \ldots, c_n)$ in $\mathbb{R}^n$ whose entries are the weights $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

We'll use the notation $[\mathbf{x}]_{\mathcal{B}}$; that is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

Different Coordinates on $\mathbb{R}^n$

We can use alternative coordinate systems to describe vectors in $\mathbb{R}^n$.

Definition

Given an ordered basis $\mathcal{B}$ in $\mathbb{R}^n$, the matrix $P_{\mathcal{B}} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n]$ is called the **change of coordinates matrix** for the basis $\mathcal{B}$ (or from the basis $\mathcal{B}$ to the standard basis).

Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis of $\mathbb{R}^n$. Then the change of coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the one to one linear transformation from $\mathbb{R}^n$ onto $\mathbb{R}^n$ defined by

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$$

$\vec{x} = P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$

Theorem: Coordinate Mapping

Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a **one to one** mapping of V **onto** $\mathbb{R}^n$.

Remark:

- For a vector space V, if there exists a coordinate mapping from $V \to \mathbb{R}^n$, we say that V is **isomorphic** to $\mathbb{R}^n$.
- Properties of subsets of V, such as linear dependence, can be discerned from the coordinate vectors in $\mathbb{R}^n$.

$\mathbb{P}_3$ is **Isomorphic** to $\mathbb{R}^4$

We saw that using the ordered basis $\mathcal{B} = \{1, t, t^2, t^3\}$ that any vector

$$\mathbf{p}(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$$

in $\mathbb{P}_3$ has coordinate vector

$$[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

in $\mathbb{R}^4$.

Example

Use coordinate vectors to determine if the set $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ is linearly dependent or independent in $\mathbb{P}_2$.

$$\mathbf{p}(t) = 1 - 2t^2, \quad \mathbf{q}(t) = 3t + t^2, \quad \mathbf{r}(t) = 1 + t$$

We need a basis for $\mathbb{P}_2$ to define the coordinate vectors. Using the basis

$\mathcal{B} = \{1, t, t^2\}$ in that order.

Then: $[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $[\vec{q}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$, $[\vec{r}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

We can use a matrix to determine if these vector in $\mathbb{R}^3$ are lin. independent.

Let $A = \begin{bmatrix} [\vec{p}]_{\mathcal{B}} & [\vec{q}]_{\mathcal{B}} & [\vec{r}]_{\mathcal{B}} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 1 \\ -2 & 1 & 0 \end{bmatrix}.$$

Using the determinant (across row 1)

$$\det(A) = 1 \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} - 0 \begin{vmatrix} \cdots \end{vmatrix} + 1 \begin{vmatrix} 0 & 3 \\ -2 & 1 \end{vmatrix}$$

$$= 1(0-1) + 1(0+6) = -1+6 = 5$$

Since $\det(A) \neq 0$, the columns of A are lin. independent.

$\{[\vec{p}]_B, [\vec{q}]_B, [\vec{r}]_B\}$ is lin. independent in $\mathbb{R}^3$.

So $\{\vec{p}, \vec{q}, \vec{r}\}$ are lin. independent in $\mathbb{P}_2$.