March 4 Math 3260 sec. 52 Spring 2024

Section 3.2: Properties of Determinants

Theorem:

Let A be an $n \times n$ matrix, and suppose the matrix B is obtained from A by performing a single elementary row operation. Then

(i) If B is obtained by adding a multiple of a row of A to another row of A (row replacement), then
$$\det(B) = \det(A).$$

(ii) If B is obtained from A by swapping any pair of rows (row swap) , then
$$\det(B) = -\det(A).$$

(iii) If B is obtained from A by scaling any row by the constant k (scaling), then
$$\det(B) = k\det(A).$$

Results on Determinants

Theorem

The $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

Theorem

For $n \times n$ matrix A, $\det(A^T) = \det(A)$.

Theorem

For $n \times n$ matrices A and B, $\det(AB) = \det(A)\det(B)$.

Remark: We showed last time that this last theorem implies $\det(A^{-1}) = (\det(A))^{-1}$ for any invertible matrix A.

Example

Let A be an $n \times n$ matrix, and suppose there exists invertible matrix P such that[1]

$$B = P^{-1}AP.$$

Show that

$$\det(B) = \det(A).$$

$$\det(B) = \det(P^{-1}AP)$$

$$= \det(P^{-1})\det(A)\det(P) \quad \leftarrow \text{product of scalars}$$

$$= \det(A)\det(P^{-1})\det(P)$$

$$= \det(A)\,(\det(P))^{-1}\det(P)$$

[1] The process of multiplying by P^{-1} on the left and P on the right is called a *similarly transform*. The matrices A and B are said to be *similar*.

$$= \det(A) \cdot 1$$

$$= \det(A).$$

Section 3.3: Cramer's Rule, Volume, and Linear Transformations

Cramer's Rule is a method for solving some small linear systems of equations.

Notation:

For $n \times n$ matrix A and $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing the i^{th} column with the vector $\mathbf{b}$. That is

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \cdots \mathbf{a}_{i-1}\ \mathbf{b}\ \mathbf{a}_{i+1} \cdots \mathbf{a}_n]$$

Example Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, then

$$A_3(\mathbf{b}) = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

Cramer's Rule

Theorem:

Let A be an $n \times n$ nonsingular matrix. Then for any vector $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution of the system $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x}$ where

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, \ldots, n$$

Remark: The condition $\det(A) \neq 0$ is necessary for Cramer's rule to be a viable method. This allows for the solution to be given in terms of ratios of determinants.

Remark: If $\det(A) = 0$, the system may be consistent, but another method is required to make a determination.

Example

Determine whether Cramer's rule can be used to solve the system. If so, use it to solve the system.

$$\begin{aligned} 2x_1 &+ x_2 &= 9 \\ -x_1 &+ 7x_2 &= -3 \end{aligned}$$

restate in the form

$A\vec{x} = \vec{b}$

$$\underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} \underbrace{=}_{=} \underbrace{\begin{bmatrix} 9 \\ -3 \end{bmatrix}}_{\vec{b}}$$

$\det(A) = 2(7) - (-1)(1) = 14 + 1 = 15$ $\quad\det(A) \neq 0$

$A_1(\vec{b}) = \begin{bmatrix} 9 & 1 \\ -3 & 7 \end{bmatrix}, \quad A_2(\vec{b}) = \begin{bmatrix} 2 & 9 \\ -1 & -3 \end{bmatrix}$

$$\det(A_1(\vec{b})) = 9(7) - (-3)(1) = 66$$

$$\det(A_2(\vec{b})) = 2(-3) - (-1)(9) = 3$$

$$x_1 = \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{66}{15} = \frac{22}{5}$$

$$x_2 = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{3}{15} = \frac{1}{5}$$

The solution is $\left(\frac{22}{5}, \frac{1}{5}\right)$.

Example

Determine whether Cramer's rule can be used to solve the system. If so, use it to solve the system.

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 3 \\ & & x_2 & + & 4x_3 & = & 3 \\ 5x_1 & + & 6x_2 & & & = & 4 \end{array}$$

restate as $A\vec{x} = \vec{b}$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}}_{\vec{b}}$$

use row 2

$$\det(A) = \overset{=0}{a_{21}} C_{21} + a_{22} C_{22} + a_{23} C_{23}$$

$$= 1(-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 5 & 0 \end{vmatrix} + 4(-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix}$$

$= (-15) \cdot 1 (6-10) = -15+16 = 1$

$$A_1(\vec{b}) = \begin{bmatrix} 3 & 2 & 3 \\ 3 & 1 & 4 \\ 4 & 6 & 0 \end{bmatrix} \qquad \det(A_1(\vec{b})) = 4\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} - 6\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix}$$

$(-1)^{2+3}$

$= 4(5) - 6(3) = 20-18 = 2$

$$A_2(\vec{b}) = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 3 & 4 \\ 5 & 4 & 0 \end{bmatrix} \qquad \det(A_2(\vec{b})) = 3\begin{vmatrix} 1 & 3 \\ 5 & 0 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 \\ 5 & 4 \end{vmatrix}$$

$= 3(-15) - 4(4-15) =$

$= -45 + 44 = -1$

$$A_3(\vec{b}) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 5 & 6 & 4 \end{bmatrix}$$

$$\det(A_3(\vec{b})) = 1\begin{vmatrix} 1 & 3 \\ 5 & 4 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix}$$

$$= (4-15) - 3(6-10)$$

$$= -11 - 3(-4) = 1$$

The solution

$$x_1 = \frac{2}{1}, \quad x_2 = \frac{-1}{1}, \quad x_3 = \frac{1}{1}$$

as a point it's $(2, -1, 1)$.

Application: Laplace Transforms

In various engineering applications, electrical or mechanical components are often chosen to try to control the long term behavior of a system (e.g. adding a damper to kill off oscillatory behavior). Using *Laplace Transforms*, differential equations are converted into algebraic equations containing a parameter s. These give rise to systems of the form

$$\begin{array}{rcrcl} 3sX & - & 2Y & = & 4 \\ -6X & + & sY & = & 1 \end{array}$$

Determine the values of s for which the system is uniquely solvable. For such s, find the solution (X, Y) using Cramer's rule.

$$
\begin{aligned}
3sX &- 2Y = 4 \\
-6X &+ sY = 1.
\end{aligned}
$$

Restate as $A\vec{x} = \vec{b}$

$$
\underbrace{\begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} X \\ Y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{\vec{b}}
$$

$\det(A) = 3s(s) - (-6)(-2) = 3s^2 - 12 = 3(s^2-4)$

The system is uniquely solvable when $\det(A) \neq 0$.

$\det(A) = 3(s^2-4) = 3(s-2)(s+2) = 0$

if $s=2$ or $s=-2$.

$\det(A) \neq 0$ for $s \neq \pm 2$.

$$\begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

For $s \neq \pm 2$,

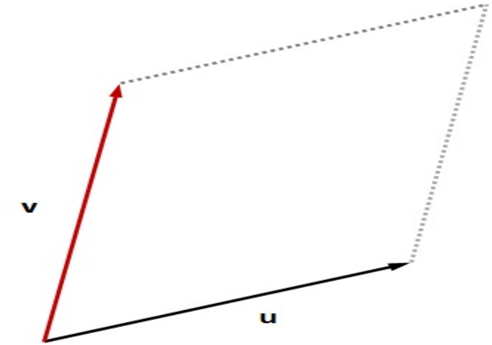
$$A_1(\vec{b}) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix} \qquad \det(A_1(\vec{b})) = 4s + 2$$

$$A_2(\vec{b}) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix} \qquad \det(A_2(\vec{b})) = 3s + 24$$

For $s \neq \pm 2$, the solution

$$X = \frac{4s+2}{3(s^2-4)}, \quad Y = \frac{3s+24}{3(s^2-4)}$$

Area & Volume (Video)

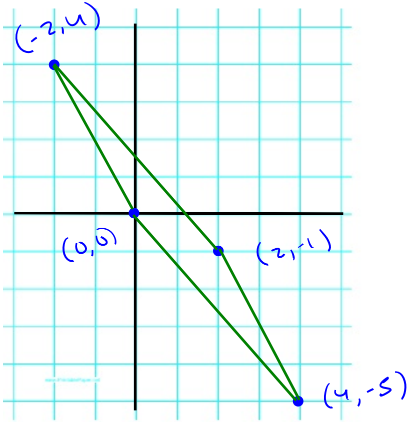


Theorem:

If **u** and **v** are nonzero, nonparallel vectors in $\mathbb{R}^2$, then the area of the parallelogram determined by these vectors is $|\det(A)|$ where $A = [\mathbf{u}\ \mathbf{v}]$.

Example

Find the area of the parallelogram with vertices $(0,0)$, $(-2,4)$, $(4,-5)$, and $(2,-1)$.



Let $\vec{u} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$

$A = \begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}$

$\det(A) = -6$

Area $= 6$