March 6 Math 3260 sec. 51 Spring 2024

Section 3.3: Cramer's Rule, Volume, and Linear Transformations

Notation:

For $n \times n$ matrix A and $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing the i^{th} column with the vector $\mathbf{b}$. That is

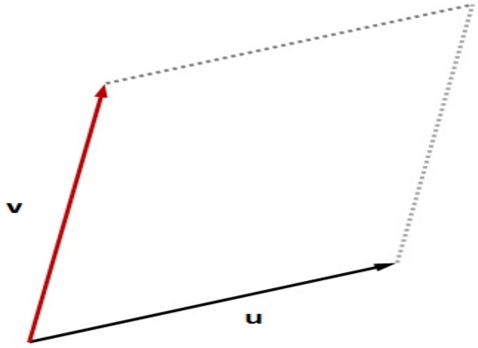
$$A_i(\mathbf{b}) = [\mathbf{a}_1 \cdots \mathbf{a}_{i-1}\ \mathbf{b}\ \mathbf{a}_{i+1} \cdots \mathbf{a}_n]$$

Theorem:

Let A be an $n \times n$ nonsingular matrix. Then for any vector $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution of the system $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x}$ where

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, \ldots, n$$

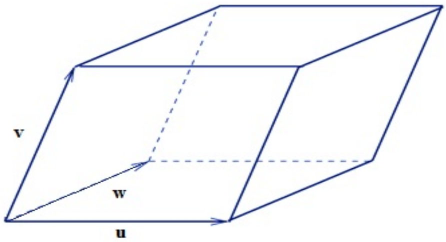
Area & Volume (Video)



Theorem:

If **u** and **v** are nonzero, nonparallel vectors in $\mathbb{R}^2$, then the area of the parallelogram determined by these vectors is $|\det(A)|$ where $A = [\mathbf{u}\ \mathbf{v}]$.

Volume of a Parallelepiped



Theorem:

If **u**, **v**, and **w** are nonzero, non-collinear vectors in $\mathbb{R}^3$, then the volume of the parallelepiped determined by these vectors is $|\det(A)|$ where $A = [\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$.

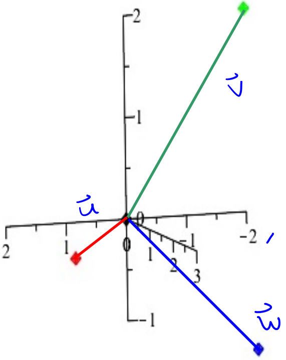
Example

Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(2, 3, 0)$, $(-2, 0, 2)$ and $(-1, 3, -1)$.

$\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$

$\vec{w} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$

Let $A = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$



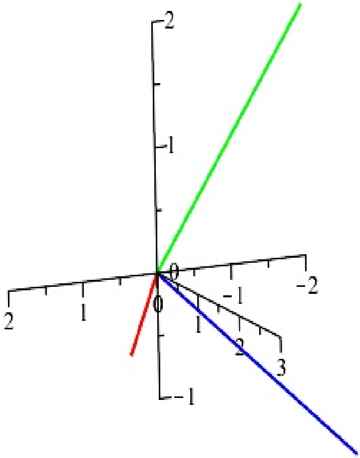
$$A = \begin{bmatrix} 2 & -2 & -1 \\ 3 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix}$$

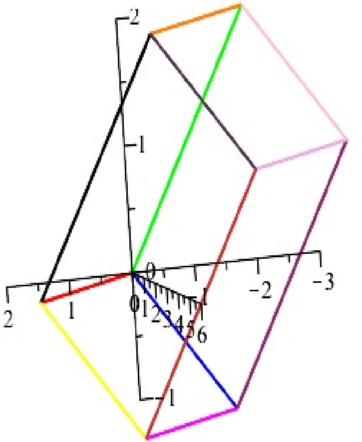
. Going down column 1

$$\det(A) = 2(-1)^{1+1}\begin{vmatrix} 0 & 3 \\ 2 & -1 \end{vmatrix} + 3(-1)^{2+1}\begin{vmatrix} -2 & -1 \\ 2 & -1 \end{vmatrix} + 0$$

$$= 2(0-6) - 3(2+2) = -12 - 12 = -24$$

$$\text{Volume} = |-24| = 24$$


2
1
0
−1
2
1
0
−1
−2
1
2
3


2
1
−1
2
1
0
−2
−3
1
2
3
4
5
6

Section 4.1: Vector Spaces and Subspaces

Recall that we had defined $\mathbb{R}^n$ as the set of all n-tuples of real numbers. We defined two operations, vector addition and scalar multiplication, and said that the following algebraic properties hold:

For every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^n$ and scalars c and d

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(vii) $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}$

We later saw that a set of $m\times n$ matrices with scalar multiplication and matrix addition satisfies the same set of properties.

Question: Are there other sets of objects with operations that share this same structure?

Definition: Vector Space

A **vector space** is a nonempty set V of objects called *vectors* together with two operations called *vector addition* and *scalar multiplication* that satisfy the following ten axioms:

For all $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V, and for any scalars c and d

1. The sum $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There exists a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each vector $\mathbf{u}$ there exists a vector $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. For each scalar c, $c\mathbf{u}$ is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = d(c\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$

Remarks:

- V is more accurately called a *real vector space* when we assume that the relevant scalars are the real numbers.
- Property 1., $\mathbf{u} + \mathbf{v} \in V$, is called being **closed** under (or *with respect to*) vector addition.
- Property 6., $c\mathbf{u} \in V$, is called being **closed** under (or *with respect to*) scalar multiplication.
- A vector space has the same basic *algebraic structure* as $\mathbb{R}^n$
- These are **axioms.** That means they are assumed, not proven. However, we can use them to prove or disprove that some set with operations is actually a vector space.

An Example of a Vector Space: "P two"

"P two"

$$\mathbb{P}_2 = \left\{ \mathbf{p}(t) = p_0 + p_1 t + p_2 t^2 \;\middle|\; p_0, p_1, p_2 \in \mathbb{R} \right\}.$$

Consider t to be some real variable, and consider the scalars to be $\mathbb{R}$. Let $\mathbb{P}_2$ be the set of all polynomials with real coefficients of degree at most two.

Examples of elements of $\mathbb{P}_2$ include things like

$$\mathbf{p}(t) = 1 + t - 3t^2, \quad \mathbf{q}(t) = -2 + 5t + 12t^2, \quad \text{and} \quad \mathbf{r}(t) = \pi + \frac{1}{\pi}t.$$

Remark: It doesn't make sense to state that $\mathbb{P}_2$ is a vector space until we define **scalar multiplication** and **vector addition**.

An Example of a Vector Space: “P two”

Let $\mathbf{p}(t) = p_0 + p_1 t + p_2 t^2$ and $\mathbf{q}(t) = q_0 + q_1 t + q_2 t^2$ be polynomials in $\mathbb{P}_2$ and c be a scalar. We define the two operations as follows:

Scalar Multiplication: $(c\mathbf{p})(t) = c\mathbf{p}(t) = cp_0 + cp_1 t + cp_2 t^2.$

Vector Addition:
$$\begin{aligned}(\mathbf{p}+\mathbf{q})(t) &= \mathbf{p}(t) + \mathbf{q}(t)\\ &= (p_0+q_0) + (p_1+q_1)t + (p_2+q_2)t^2.\end{aligned}$$

Remark: It can be shown that $\mathbb{P}_2$ with these operations satisfies the ten vector space axiom. Note, this means that

the polynomials ARE vectors.