March 8 Math 3260 sec. 51 Spring 2024

Section 4.1: Vector Spaces and Subspaces

Definition: Vector Space

A **vector space** is a nonempty set V of objects called *vectors* together with two operations called *vector addition* and *scalar multiplication* that satisfy the following ten axioms:

For all $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V, and for any scalars c and d

1. The sum $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There exists a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each vector $\mathbf{u}$ there exists a vector $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. For each scalar c, $c\mathbf{u}$ is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$

An Example of a Vector Space: “P two”

“P two”

$$\mathbb{P}_2 = \left\{ \mathbf{p}(t) = p_0 + p_1 t + p_2 t^2 \;\middle|\; p_0, p_1, p_2 \in \mathbb{R} \right\}.$$

Consider t to be some real variable, and consider the scalars to be $\mathbb{R}$. Let $\mathbb{P}_2$ be the set of all polynomials with real coefficients of degree at most two.

Examples of elements of $\mathbb{P}_2$ include things like

$$\mathbf{p}(t) = 1 + t - 3t^2, \quad \mathbf{q}(t) = -2 + 5t + 12t^2, \quad \text{and} \quad \mathbf{r}(t) = \pi + \frac{1}{\pi}t.$$

Remark: It doesn't make sense to state that $\mathbb{P}_2$ is a vector space until we define **scalar multiplication** and **vector addition**.

An Example of a Vector Space: “P two”

Let $\mathbf{p}(t) = p_0 + p_1 t + p_2 t^2$ and $\mathbf{q}(t) = q_0 + q_1 t + q_2 t^2$ be polynomials in $\mathbb{P}_2$ and c be a scalar. We define the two operations as follows:

Scalar Multiplication: $\quad (c\mathbf{p})(t) = c\mathbf{p}(t) = cp_0 + cp_1 t + cp_2 t^2.$

Vector Addition:
$$\begin{aligned}(\mathbf{p}+\mathbf{q})(t) &= \mathbf{p}(t) + \mathbf{q}(t)\\ &= (p_0+q_0) + (p_1+q_1)t + (p_2+q_2)t^2.\end{aligned}$$

Remark: It can be shown that $\mathbb{P}_2$ with these operations satisfies the ten vector space axiom. Note, this means that

the polynomials ARE vectors.

Example

$$\mathbf{p}(t) = 1 + t - 3t^2, \quad \mathbf{q}(t) = -2 + 5t + 12t^2, \quad \text{and} \quad \mathbf{r}(t) = \pi + \frac{1}{\pi}t.$$

Evaluate

1. $(\mathbf{p} + \mathbf{q})(t) = \vec{p}(t) + \vec{q}(t) = (1-2) + (1+5)t + (-3+12)t^2$
   $= -1 + 6t + 9t^2$

2. $(-1\mathbf{r})(t) = -1\vec{r}(t) = -1(\pi) + (-1)\frac{1}{\pi}t = -\pi - \frac{1}{\pi}t$

3. $(-1\mathbf{r} + \mathbf{r})(t) = -1\vec{r}(t) + \vec{r}(t) = -\pi + \pi + \left(-\frac{1}{\pi} + \frac{1}{\pi}\right)t$
   $= 0 + 0t$

The Zero Vector in $\mathbb{P}_2$

Let $\mathbf{0}(t) = a_0 + a_1 t + a_2 t^2$ be the zero vector in $\mathbb{P}_2$.

Use the property[1] in Axiom 4 to identify the values of the coefficients a_0, a_1, and a_2.

Let $\vec{p}(t) = p_0 + p_1 t + p_2 t^2$

$$(\vec{0} + \vec{p})(t) = \vec{0}(t) + \vec{p}(t) = \vec{p}(t)$$

$$= (a_0 + p_0) + (a_1 + p_1) t + (a_2 + p_2) t^2$$

$$= p_0 + p_1 t + p_2 t^2$$

$$\Rightarrow \quad a_0 + p_0 = p_0 \quad \Rightarrow \quad a_0 = 0$$

[1] Axiom 4 says that $\mathbf{p} + \mathbf{0} = \mathbf{p}$ for every vector $\mathbf{p}$ in $\mathbb{P}_2$.

$$a_1 + p_1 = p_1 \quad \Rightarrow \quad a_1 = 0$$

$$a_2 + p_2 = p_2 \quad \Rightarrow \quad a_2 = 0$$

That is, $a_0 = a_1 = a_2 = 0$ so

$$\vec{0}(t) = 0 + 0t + 0t^2$$

$\mathbb{P}_n$

For an integer $n \geq 0$, let $\mathbb{P}_n$ denote the set of all polynomials with real coefficients of degree at most n.

$$\mathbb{P}_n = \left\{ \mathbf{p}(t) = p_0 + p_1 t + \cdots + p_n t^n \;\middle|\; p_0, p_1, \ldots, p_n \in \mathbb{R} \right\},$$

For $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{P}_n$ and scalar c, define scalar multiplication and vector addition by

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = cp_0 + cp_1 t + \cdots + cp_n t^n,$$

$$(\mathbf{p}+\mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t) = (p_0 + q_0) + (p_1 + q_1)t + \cdots + (p_n + q_n)t^n.$$

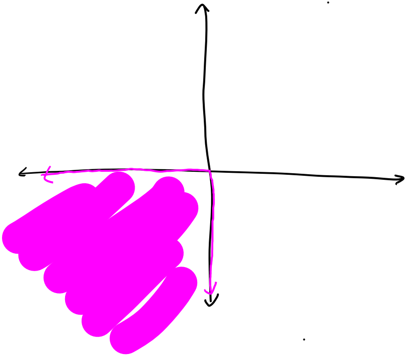
Remark: It can readily be shown that the zero vector $\mathbf{0}(t) = 0 + 0t + \cdots + 0t^n$, and the additive inverse of $\mathbf{p}(t) = p_0 + p_1 t + \cdots + p_n t^n$ is

$$-\mathbf{p}(t) = -p_0 - p_1 t - \cdots - p_n t^n.$$

A set that is not a Vector Space

Let $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, x \leq 0,\ y \leq 0 \right\}$ with regular vector addition and scalar multiplication in $\mathbb{R}^2$.

Geometrically, what is the set S?



S is the third quadrant including the negative x and y axes.

A set that is not a Vector Space

$S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, x \leq 0,\ y \leq 0 \right\}$ with regular vector addition and scalar multiplication in $\mathbb{R}^2$.

Does Axiom[2] 1. hold for S?

Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{u} = \begin{bmatrix} u \\ v \end{bmatrix}$ in S.

Then $x \leq 0$, $y \leq 0$, $u \leq 0$, and $v \leq 0$.

$\vec{x} + \vec{u} = \begin{bmatrix} x+u \\ y+v \end{bmatrix}$

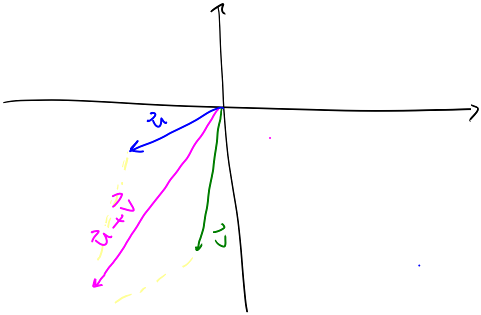
$x+u \leq 0$ and

$y+v \leq 0$

$\Rightarrow \vec{x} + \vec{u} \in S$.

[2] Axiom 1 is closure under vector addition.

S is closed under vector addition.



A set that is not a Vector Space

$S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x \leq 0, \; y \leq 0 \right\}$ with regular vector addition and scalar multiplication in $\mathbb{R}^2$.

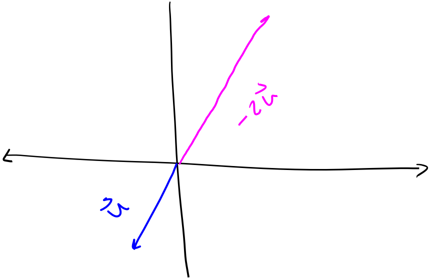
Does Axiom[3] 6. hold for S?

Consider $\vec{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$. $\vec{u}$ is in S.

$-2\vec{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$: $-2\vec{u} \notin S$

S is not closed under scalar mult.

[3] Axiom 6 is closure under scalar multiplication.



$-2\vec{v}$
$\vec{v}$

Some Algebraic Properties of Vector Spaces

Theorem

Let V be a vector space. For each $\mathbf{u}$ in V and scalar c

(i) $0\mathbf{u} = \mathbf{0}$

(ii) $c\mathbf{0} = \mathbf{0}$

(iii) $-1\mathbf{u} = -\mathbf{u}$

Proof of (i).

Let $\vec{u} \in V$ be arbitrary. Note that the scalar $0 = 0 + 0$.

$$0\vec{u} = (0+0)\vec{u}$$

$$= 0\vec{u} + 0\vec{u}$$

There exist a vector $-0\vec{u}$. Add this to both sides.

$$-0\vec{u} + 0\vec{u} = -0\vec{u} + 0\vec{u} + 0\vec{u}$$

$$\vec{0} = \vec{0} + 0\vec{u}$$

$$\vec{0} = 0\vec{u}.$$

This is the required conclusion.

Subspaces

Definition:

A **subspace** of a vector space V is a subset H of V for which

a) The zero vector is in[a] H

b) H is closed under vector addition. (i.e. $\mathbf{u}, \mathbf{v}$ in H implies $\mathbf{u} + \mathbf{v}$ is in H)

c) H is closed under scalar multiplication. (i.e. $\mathbf{u}$ in H implies $c\mathbf{u}$ is in H)

[a] This is sometimes replaced with the condition that H is nonempty.

Remark: A subspace is a vector space. If these three properties hold, it inherits the structure from its parent space.

Example

Determine which of the following is a subspace of $\mathbb{R}^2$.

1. The set of all vectors of the form $\mathbf{u} = (u_1, 0)$.

Call the set H.

Is $\vec{0}$ in H? Does $(0,0) = (u_1, 0)$ for some real u_1? Yes, if $u_1 = 0$, then we get $(0,0)$.

If $\vec{u}, \vec{v} \in H$ is $\vec{u} + \vec{v} \in H$?

$\vec{u} = (u_1, 0)$, $\vec{v} = (v_1, 0)$.

$\vec{u} + \vec{v} = (u_1 + v_1,\ 0+0) = (u_1 + v_1, 0)$.

Yes, $\vec{u} + \vec{v} \in H$

Is $c\vec{u} \in H$. $c\vec{u} = (cu_1, c(0)) = (cu_1, 0)$

Yes. H is closed under scalar multiplication.

H is a subspace of $\mathbb{R}^2$.