May 18 Math 2254 sec 001 Summer 2015

Section 4.8: Antiderivatives

Definition: A function F is called an antiderivative of f on an interval I if

$$F'(x) = f(x) \quad \text{for all } x \text{ in } I.$$

For Example: $F(x) = x^2$ is an antiderivative of $f(x) = 2x$ on $(-\infty, \infty)$.

Show that $F(x) = x^2 - 3$ is also an antiderivative of $f(x) = 2x$ on $(-\infty, \infty)$.

$$F'(x) = \frac{d}{dx}(x^2 - 3) = 2x + 0 = 2x = f(x)$$

Most General Antiderivative

Theorem: If F is any antiderivative of f on an interval I, then the *most general* antiderivative of f on I is

$$F(x) + C \quad \text{where } C \text{ is an arbitrary constant.}$$

Corollary: If $f'(x) = g'(x)$ for each x in an interval I, then $f(x) = g(x) + C$ on I.

Find the most general antiderivative of f.

(a) $f(x) = \cos x \quad I = (-\infty, \infty)$

$$F(x) = \sin x + C$$

(b) $f(x) = \sec x \tan x \quad I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$F(x) = \sec x + C$$

Find the most general antiderivative of f.

(c) $f(x) = \dfrac{1}{x^3}$ $\quad I = (0, \infty)$

$f(x) = x^{-3}$

If $g(x) = x^{-2}$

$g'(x) = -2x^{-3}$

$\Rightarrow -\frac{1}{2} g'(x) = x^{-3}$

$F(x) = \dfrac{-x^{-2}}{2} + C$

(d) $f(x) = 2^x$ $\quad I = (-\infty, \infty)$

If $g(x) = 2^x$ then

$g'(x) = 2^x \ln 2$

$\dfrac{1}{\ln 2} g'(x) = 2^x$

$F(x) = \dfrac{2^x}{\ln 2} + C$

Find the most general antiderivative of

$$f(x) = x^n, \quad \text{where } n = 1, 2, 3, \ldots$$

$g(x) = x^{n+1}$, $g'(x) = (n+1)\, x^{n+1-1}$

$= (n+1)\, x^n$

$$\boxed{F(x) = \frac{x^{n+1}}{n+1} + C}$$

$\Rightarrow \frac{1}{n+1}\, g'(x) = x^n$

Some general results[1]:

Function	Particular Antiderivative	Function	Particular Antiderivative
$cf(x)$	$cF(x)$	$\cos x$	$\sin x$
$f(x)+g(x)$	$F(x)+G(x)$	$\sin x$	$-\cos x$
$x^n,\ n\neq -1$	$\frac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
x^{-1}	$\ln\lvert x\rvert$	a^x	$\frac{1}{\ln(a)}a^x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$	$\frac{1}{1+x^2}$	$\tan^{-1} x$

A more complete table is found on page 330 in Sullivan and Miranda.

[1] We'll use the term particular antiderivative to refer to any antiderivative that has no arbitrary constant in it.

Example

Find the most general antiderivative of

(a) $s(x) = \dfrac{1}{\sqrt{1-x^2}} + \dfrac{1}{x}$

$$S(x) = \sin^{-1} x + \ln|x| + C$$

(b) $h(x) = x\sqrt{x}$ $= x \cdot x^{1/2} = x^{1+\frac{1}{2}} = x^{3/2}$

$$H(x) = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C = \frac{2}{5} x^{5/2} + C$$

Example

Determine the function $H(x)$ that satisfies the following conditions

$$H'(x) = x\sqrt{x}, \quad \text{for all } x > 0, \text{ and } \; H(1) = 0.$$

From the last example

$$H(x) = \frac{2}{5} x^{5/2} + C$$

$$H(1) = \frac{2}{5}(1)^{5/2} + C = 0 \Rightarrow \quad \frac{2}{5} + C = 0$$

$$\Rightarrow \quad C = \frac{-2}{5}$$

So $\boxed{H(x) = \frac{2}{5} x^{5/2} - \frac{2}{5}}$

Example

A particle moves along the x-axis so that its acceleration at time t is given by

$$a(t) = 12t - 2 \quad \text{m/sec}^2.$$

At time $t = 0$, the velocity v and position s of the particle are known to be

$$v(0) = 3 \quad \text{m/sec, and} \qquad s(0) = 4 \text{ m.}$$

Find the position $s(t)$ of the particle for all $t > 0$.

$a(t) = \frac{dv}{dt}$ $\qquad$ $v(t) = 12\frac{t^2}{2} - 2t + C$

$v(t) = 6t^2 - 2t + C$ $\qquad$ $v(0) = 6\cdot 0^2 - 2\cdot 0 + C = 3$

$\Rightarrow C = 3$

$$v(t) = 6t^2 - 2t + 3 \qquad v = \frac{ds}{dt}$$

$$s(t) = 6\frac{t^3}{3} - 2\frac{t^2}{2} + 3t + C$$

$$s(t) = 2t^3 - t^2 + 3t + C$$

$$s(0) = 2\cdot 0^3 - 0^2 + 3\cdot 0 + C = 4 \Rightarrow C = 4$$

The position for $t > 0$ is

$$\boxed{s(t) = 2t^3 - t^2 + 3t + 4}$$

Example

A **differential equation** is an equation that involves the derivative(s) of an unknown function. **Solving** such an equation would mean finding such an unknown function.

Solve the differential equation subject to the given *initial* conditions.

$$\frac{d^2y}{dx^2} = \cos x + 2, \quad y(0) = 0, \quad y'(0) = -1$$

$\frac{dy}{dx} = \sin x + 2x + C$ $y'(0) = \sin(0) + 2\cdot 0 + C = -1$

$\Rightarrow C = -1$

$\frac{dy}{dx} = \sin x + 2x - 1$

$$y = -\cos x + x^2 - x + C$$

$$y(0) = -\cos(0) + 0^2 - 0 + C = 0$$

$$-1 + C = 0 \Rightarrow C = 1$$

$$y(x) = -\cos x + x^2 - x + 1$$

Section 5.2: The Definite Integral

We saw that a sum of the form

$$f(u_1^*)\Delta x + f(u_2^*)\Delta x + \cdots + f(u_n^*)\Delta x$$

approximated the area of a region if f was continuous and positive. And that under these conditions, the limit

$$\lim_{n\to\infty}\sum_{i=1}^n f(u_i^*)\Delta x = \lim_{n\to\infty}[f(u_1^*)\Delta x + f(u_2^*)\Delta x + \cdots + f(u_n^*)\Delta x]$$

was the value of this area.

Can we generalize this dropping the requirement that f is positive? that f is continuous?

Some Terminology to Recall

- A **Partition** P of an interval $[a, b]$ is a collection of points $\{x_0, x_1, ..., x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

- A **Subinterval** is one of the intervals $x_{i-1} \le x \le x_i$ determined by a partition.
- The width of a subinterval is denoted $\Delta x_i = x_i - x_{i-1}$. If they are all the same size (equal spacing), then

$$\Delta x = \frac{b-a}{n}, \quad \text{and this is called the } \textbf{norm} \text{ of the partition.}$$

- A set of **sample points** is a set $\{u_1^*, u_2^*, ..., u_n^*\}$ such that $x_{i-1} \le u_i^* \le x_i$.

Taking the number of rectangles to ∞ is the same as taking the width $\Delta x \to 0$.

Definition (Definite Integral)

Let f be defined on an interval $[a, b]$. Let

$$x_0 = a < x_1 < x_2 < \cdots < x_n = b$$

be any partition of $[a, b]$, and $\{u_1^*, u_2^*, \ldots, u_n^*\}$ be any set of sample points. Then the **definite integral of f from a to b** is denoted and defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(u_i^*)\Delta x_i$$

provided this limit exists. Here, the limit is taken over all possible partitions of $[a, b]$.

Terms and Notation

- **Riemann Sum:** any sum of the form $f(u_1^*)\Delta x + f(u_2^*)\Delta x + \cdots + f(u_n^*)\Delta x$
- **Integral Symbol/Sign:** $\int$ (a stretched "S" for "sum")
- **Integrable:** If the limit does exists, f is said to be integrable on $[a, b]$
- **Limits of Integration:** a is called the lower limit of integration, and b is the upper limit of integration
- **Integrand:** the expression "$f(x)$" is called the integrand

- **Differential:** dx is called a differential, it indicates what the variable is and can be thought of as the limit of Δx (just as it is in the derivative notation "$\frac{dy}{dx}$").

- **Dummy Variable/Variable of Integration:** the variable that appears in both the integrand and in the differential. For example, if the differential is dx, the dummy variable is x; it the differential is du, the dummy variable is u

$$\int_a^b f(x)\,dx$$

Important Remarks

(1) If the integral does exist, it is a **number**. That is, it does not depend on the dummy variable of integration. The following are equivalent.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(q)\,dq$$

(2) The definite integral is a limit of Riemann Sums!

(3) If f is positive and continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \text{ the area under the curve.}$$

What if f is continuous, but not always positive?

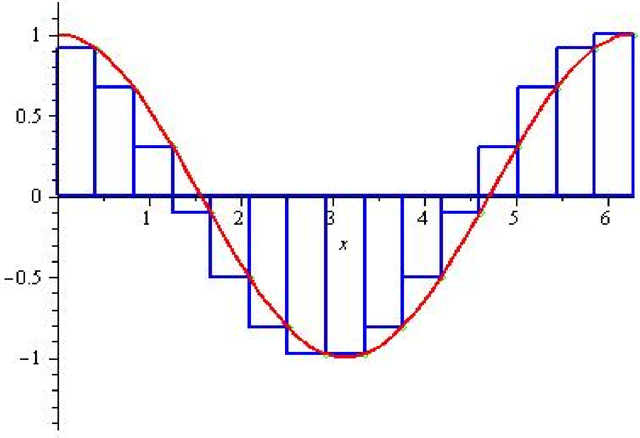


Figure: A function that changes signs on $[a, b]$. (Here, $f(x) = \cos x$, $a = 0$ and $b = 2\pi$; the partition has 15 subintervals.)

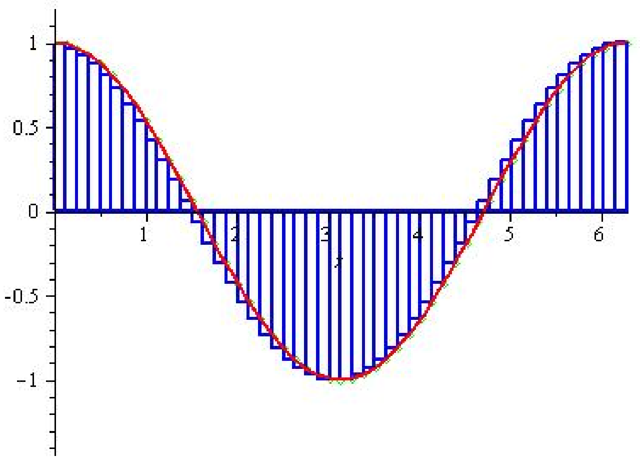


Figure: The same function but with 50 subintervals.

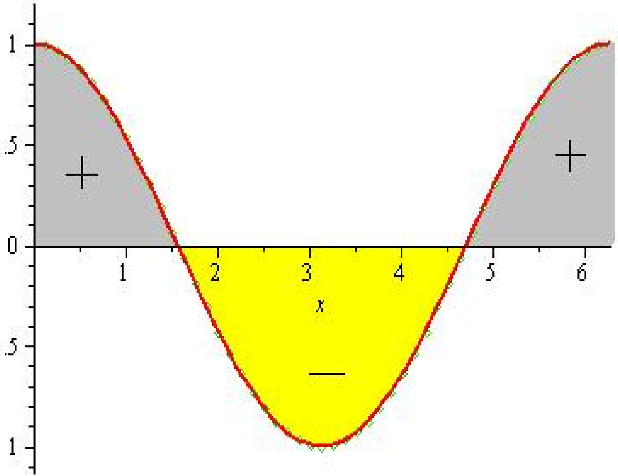
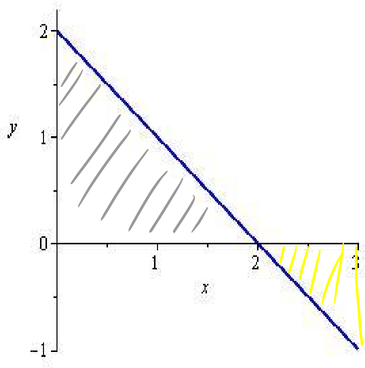
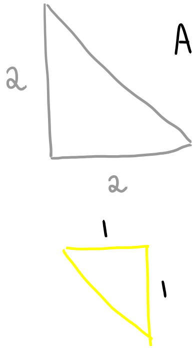


Figure: $\int_a^b f(x)\,dx =$ area of gray region $-$ area of yellow region

Example

Use area to evaluate the integral $\int_0^3 (2 - x)\,dx$.





Area $= \frac{1}{2}(2)(2) = 2$

Area $= \frac{1}{2}(1)(1) = \frac{1}{2}$

$$\int_0^3 (2-x)\,dx = 2 - \frac{1}{2} = \frac{3}{2}$$

Important Theorems:

Theorem: If f is continuous on $[a, b]$ or has only finitely many jump discontinuities on $[a, b]$, then f is integrable on $[a, b]$

Theorem: If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(u_i)\Delta x,$$

where

$$\Delta x = \frac{b-a}{n}, \quad \text{and} \quad u_i = a + i\Delta x.$$

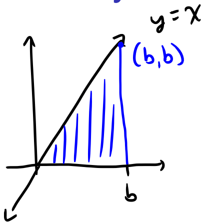
Examples:

$$\int_0^{2\pi} \cos x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \cos\left(\frac{2\pi i}{n}\right)\frac{2\pi}{n}$$

$$\int_2^4 \sqrt{t}\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{2+\frac{2i}{n}}\left(\frac{2}{n}\right)$$

Show that $\int_0^b x\,dx = \frac{b^2}{2}$ by using (i) a Riemann sum[2] and (ii) geometry.

ii)



$$\text{Area} = \frac{1}{2}(b)(b) = \frac{b^2}{2}$$

$$\int_0^b x\,dx = \frac{b^2}{2}$$

i) Riemann Sum:

$$\Delta x = \frac{b-0}{n} = \frac{b}{n} \qquad f(x) = x$$

$$u_i^* = 0 + i\,\Delta x = i\,\frac{b}{n}$$

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[2] The following identity is useful

$$\sum_{i=1}^{n} i = \frac{n^2+n}{2},$$

$$\int_0^b x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(u_i^*)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} i\left(\frac{b}{n}\right)\frac{b}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} i\,\frac{b^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{b^2}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{b^2}{n^2}\left(\frac{n^2+n}{2}\right)$$

$$= \lim_{n\to\infty} b^2\left(\frac{\cancel{n}(n+1)}{2n^{\cancel{2}}}\right)$$

$$= \lim_{n\to\infty} b^2\left(\frac{n+1}{2n}\right)$$

$$= \lim_{n\to\infty} b^2\left(\frac{1}{2}+\frac{1}{2n}\right) = b^2\left(\frac{1}{2}+0\right) = \frac{b^2}{2}$$

Some First Properties of Definite Integrals

Suppose that f and g are integable on $[a, b]$ and let c be constant.

(1) $\displaystyle\int_a^b c\,dx = c(b-a)$

(2) $\displaystyle\int_a^a f(x)\,dx = 0,\quad$ if $f(a)$ exists

(3) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

Example

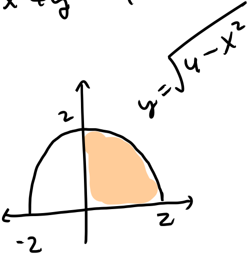
Evaluate the integral using areas and the properties.

$$\int_2^0 \sqrt{4-x^2}\,dx = -\int_0^2 \sqrt{4-x^2}\,dx$$

$$y = \sqrt{4-x^2} \Rightarrow y^2 = 4-x^2 \Rightarrow x^2+y^2=4$$

$$\text{Area} = \frac{1}{4}\pi(2)^2$$

$$= \pi$$



$$\int_2^0 \sqrt{4-x^2}\,dx = -\pi$$