May 19 Math 2254 sec 001 Summer 2015

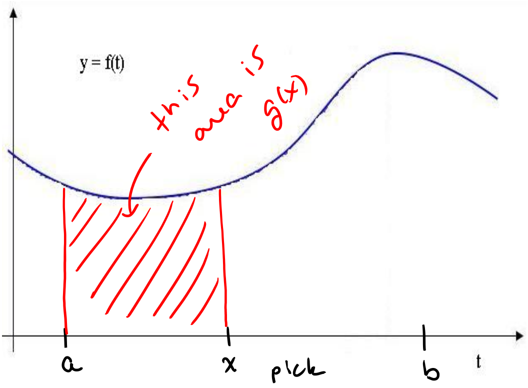
Section 5.3: The Fundamental Theorem of Calculus

Suppose f is continuous on the interval $[a, b]$. For $a \le x \le b$ define a new function

$$g(x) = \int_a^x f(t)\,dt$$

How can we understand this function, and what can be said about it?

Geometric interpretation of $g(x) = \int_a^x f(t)\,dt$



Figure

FTC Applet 1 FTC Applet 2

Theorem: The Fundamental Theorem of Calculus (part 1)

If f is continuous on $[a, b]$ and the function g is defined by

$$g(x) = \int_a^x f(t)\,dt \quad \text{for} \quad a \le x \le b,$$

then g is continuous on $[a, b]$ and differentiable on (a, b). Moreover

$$g'(x) = f(x).$$

i.e.

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

This means that the new function g is an **antiderivative** of f on (a, b)!

"FTC" = "fundamental theorem of calculus"

Example:

Evaluate each derivative.

(a) $\displaystyle \frac{d}{dx}\int_0^x \sin^2(t)\,dt$

here $\sin^2(t) = f(t)$

$= \sin^2(x)$

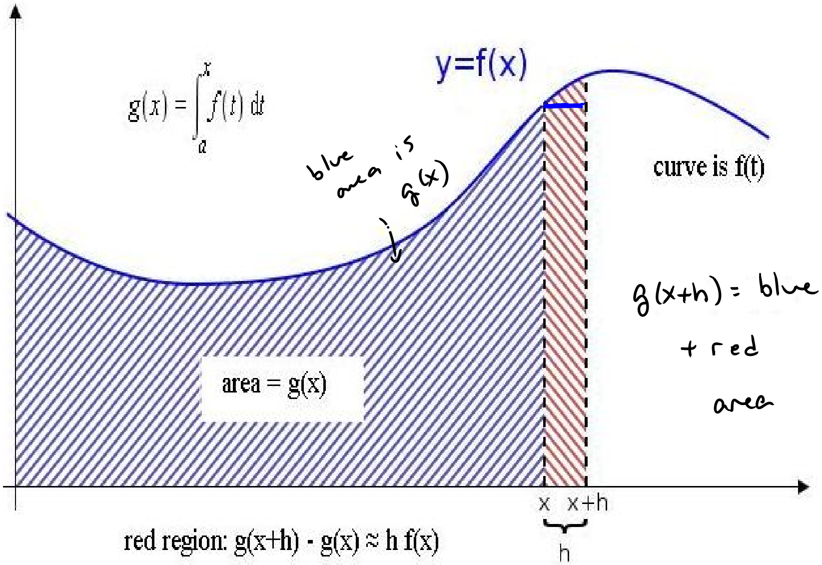
(b) $\displaystyle \frac{d}{dx}\int_4^x \frac{t-\cos t}{t^4+1}\,dt$

here $f(t) = \dfrac{t - \cos t}{t^4 + 1}$

$= \dfrac{x - \cos x}{x^4 + 1}$

Geometric Argument of FTC

$$\frac{d}{dx} g(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$



$g(x+h) - g(x) =$ blue + red - blue = red area

$\approx$ area of rectangle $= f(x)\,h$

$$g(x+h) - g(x) \approx f(x)\,h$$

$$\frac{g(x+h) - g(x)}{h} \approx f(x)$$ take $h \to 0$

$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = g'(x) = f(x)$$

Chain Rule with FTC

Evaluate each derivative.

(a) $\dfrac{d}{dx}\displaystyle\int_0^{x^2} t^3\,dt$

Let $u = x^2 \rightarrow \dfrac{du}{dx} = 2x$

Then $\displaystyle\int_0^{x^2} t^3\,dt = \int_0^u t^3\,dt$

$$= \left(\frac{d}{du}\int_0^u t^3\,dt\right)\frac{du}{dx}$$

Recall

$$\frac{d}{dx}F(u) = \frac{dF}{du}\cdot\frac{du}{dx}$$

$$= u^3 \cdot 2x$$

$$= (x^2)^3\cdot(2x) = x^6(2x) = 2x^7$$

(b) $\frac{d}{dx}\int_x^7 \cos(t^2)\,dt = \frac{d}{dx}\left(-\int_7^x \cos(t^2)\,dt\right)$

$$= -\frac{d}{dx}\int_7^x \cos(t^2)\,dt$$

$$= -\cos(x^2)$$

Leibniz Rule

Suppose a and b are differentiable functions and f is continuous.

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))b'(x) - f(a(x))a'(x)$$

Example:

$$\frac{d}{dt}\int_{x^2}^{\sqrt{x}} f(t)\,dt = f(\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right) - f(x^2)(2x) = \frac{f(\sqrt{x})}{2\sqrt{x}} - 2xf(x^2).$$

Theorem: The Fundamental Theorem of Calculus (part 2)

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f on $[a, b]$. (i.e. $F'(x) = f(x)$)

Example: Use the FTC to show that $\int_0^b x\,dx = \frac{b^2}{2}$

Here $f(x) = x$ an antiderivative is

$$F(x) = \frac{x^2}{2}$$

$$\int_0^b x\,dx = F(b) - F(0) = \frac{b^2}{2} - \frac{0^2}{2}$$

$$= \frac{b^2}{2}$$

Notation

Suppose F is an antiderivative of f. We write

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

or sometimes

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

For example

$$\int_0^b x\ dx = \frac{x^2}{2}\Big|_0^b = \frac{b^2}{2} - \frac{0^2}{2} = \frac{b^2}{2}$$

Evaluate each definite integral using the FTC

(a) $\int_0^2 3x^2\,dx$ $= x^3 \Big|_0^2$

$= 2^3 - 0^3 = 8$

(b) $\int_{-3}^{-1} \frac{1}{x}\,dx = \ln|x| \Big|_{-3}^{-1} = \ln|-1| - \ln|-3|$

$= \ln(1) - \ln(3)$

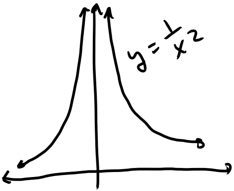
$= 0 - \ln 3 = -\ln 3$

Caveat! The FTC doesn't apply if f is not continuous!

The function $f(x) = \frac{1}{x^2}$ is positive everywhere on its domain. Now consider the calculation

$$\int_{-1}^{2} \frac{1}{x^2}\,dx = \left.\frac{x^{-1}}{-1}\right|_{-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}$$

Is this believable? Why or why not?



area here can't be negative!!

Section 5.4: Properties of Definite Integrals

Suppose that f and g are integable on $[a, b]$ and let c be constant.

(1) $$\int_a^b c\,dx = c(b-a)$$

(2) $$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

(3) $$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Properties of Definite Integrals Continued...

(4) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

(5) $\displaystyle\int_a^a f(x)\,dx = 0$

(6) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Properties of Definite Integrals Continued...

(7) If $f(x) \leq g(x)$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

(8) And, as an immediate consequence of (7) and (1), if $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Example

Evaluate $\displaystyle\int_0^1 \left(4x - \frac{3}{1+x^2}\right) dx$

$$= \int_0^1 4x \, dx - \int_0^1 \frac{3}{1+x^2} \, dx$$

$$= 4 \int_0^1 x \, dx - 3 \int_0^1 \frac{1}{1+x^2} \, dx$$

$$= 4 \left. \frac{x^2}{2} \right|_0^1 - 3 \tan^{-1} x \Big|_0^1 = 2(1)^2 - 2(0)^2 - \left[3 \tan^{-1} 1 - 3 \tan^{-1} 0\right]$$

$$= 2 - \frac{3\pi}{4}$$

Example

Show that property (8) guarantees that

$$0 \leq \int_0^2 xe^{-x}\,dx \leq \frac{2}{e}$$

Here $f(x) = xe^{-x}$, $a=0$ and $b=2$. We need to find m, M such that $m \leq xe^{-x} \leq M$ on $[0,2]$

Find the abs. max and min of f:

Find critical #

$$f'(x) = 1 \cdot e^{-x} + x \cdot e^{-x}(-1) = e^{-x} - xe^{-x}$$

$$f'(x) = 0 \Rightarrow e^{-x} - xe^{-x} = 0$$

$$\Rightarrow e^{-x}(1-x) = 0 \Rightarrow \begin{array}{l} e^{-x} = 0 \text{ or} \\ 1 - x = 0 \end{array}$$

$e^{-x} \neq 0$ for all x so there is one critical # $x = 1$.

Check ends and critical #

$f(0) = 0e^{-0} = 0$ ← minimum

$f(1) = 1e^{-1} = e^{-1}$ ← maximum

$f(2) = 2e^{-2} = \frac{2}{e^2}$ $\quad \frac{2}{e^2} < \frac{2}{2e} = \frac{1}{e}$

So $m = 0$ and $M = \frac{1}{e} = e^{-1}$

$$0(2-0) \le \int_0^2 x e^{-x}\, dx \le \frac{1}{e}(2-0)$$

$$0 \le \int_0^2 x e^{-x}\, dx \le \frac{2}{e}$$

as required.

Average Value of a Function

For a finite collection of numbers $y_1, y_2, \ldots, y_n$, the average (arithmetic) value is the number

$$y_{avg} = \frac{y_1 + y_2 + \cdots + y_n}{n}.$$

We'd like to extend this notation to an infinite collection of numbers $y = f(x)$ for $a \le x \le b$.

If we take a set of sample points $u_1^*, u_2^*, \ldots, u_n^*$ for an equally spaced partition of $[a, b]$, we could approximate

$$y_{avg} \approx \frac{f(u_1^*) + f(u_2^*) + \cdots + f(u_n^*)}{n}.$$

$$y_{avg} \approx \frac{f(u_1^*) + f(u_2^*) + \cdots + f(u_n^*)}{n}.$$

For an equally spaced partition

$$\Delta x = \frac{b-a}{n} \quad \Longrightarrow \quad \frac{1}{n} = \frac{\Delta x}{b-a}.$$

So replacing n we can write

$$y_{avg} \approx \sum_{i=1}^{n} f(u_i^*) \frac{\Delta x}{b-a} = \frac{1}{b-a} \sum_{i=1}^{n} f(u_i^*) \Delta x.$$

We will define the average value of f on the interval $[a, b]$ as the limit of this approximation when $n \to \infty$.

Average Value of a function f on an interval $[a, b]$.

Definition: Provided f is integrable on $[a, b]$, the average value of f on $[a, b]$ is

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number u in (a, b) such that

$$f(u) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

In other words, $$\int_a^b f(x)\,dx = f(u)(b-a).$$

MVT for Integrals

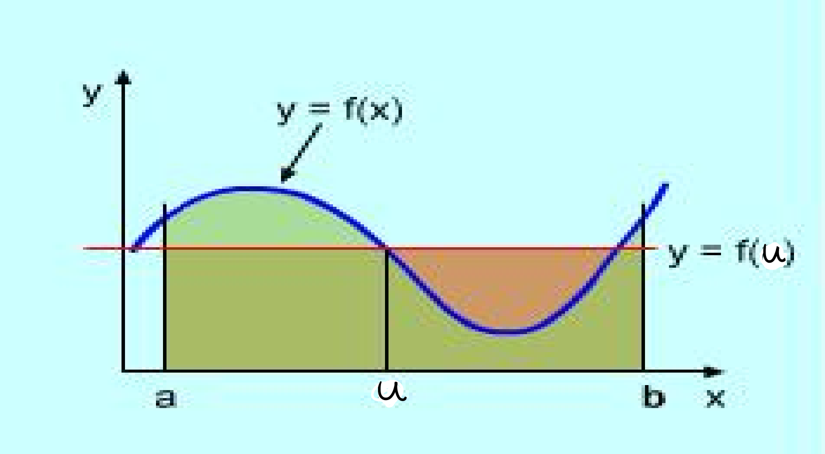


Figure: Mean Value Theorem Illustrated.

Find the average value of $f(x) = \sqrt{x}$ on $[0, 4]$.

Then find the value of u that satifies the conclusion of the MVT for integrals.

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

$$f_{avg} = \frac{1}{4-0} \int_0^4 \sqrt{x}\,dx = \frac{1}{4} \int_0^4 x^{1/2}\,dx$$

$$= \frac{1}{4} \left. \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right|_0^4 = \frac{1}{4} \left. \frac{x^{3/2}}{3/2} \right|_0^4$$

$$= \frac{1}{4} \frac{2}{3} \left. x^{3/2} \right|_0^4 = \frac{1}{6} \left(4^{3/2} - 0^{3/2} \right) = \frac{8}{6} = \frac{4}{3}$$

$$f_{avg} = \frac{4}{3}$$

The MVT says $f(u) = f_{avg}$ for some u in $(0,4)$

$$f(u) = \sqrt{u} = \frac{4}{3}$$

$$u = \frac{16}{9}$$

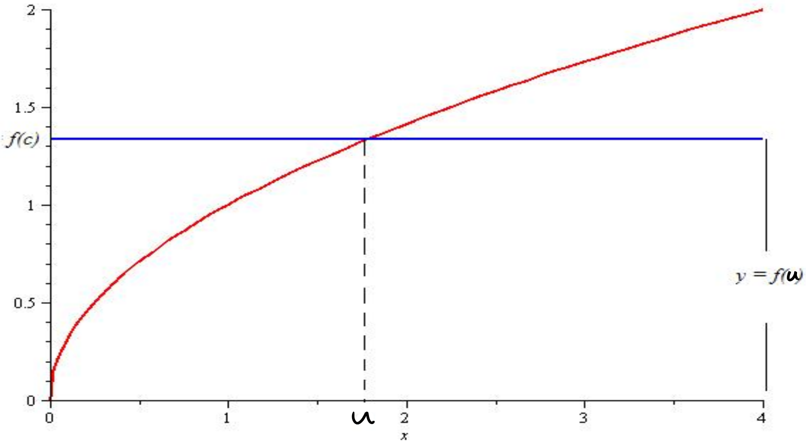


Figure: Mean Value Theorem Illustrated.

Section 5.5: The Indefinite Integral

New notation for antiderivatives:

If $F'(x) = f(x)$, i.e. F is any antiderivative of f, we will write

$$\int f(x)\,dx = F(x) + C$$

and we'll call $\int f(x)\,dx$ the **indefinite** integral of f.

For example:

$$\int 2x\,dx = x^2 + C, \quad \text{and} \quad \int \cos t\,dt = \sin t + C$$

Examples

$$\frac{d}{dt} -e^{-t} = e^{-t}$$

(a) Evaluate $\displaystyle\int e^{-t}\,dt$

$$= -e^{-t} + C$$

(b) $\displaystyle\int -\frac{\cos x}{\sin^2 x}\,dx$

Note

$$\frac{\cos x}{\sin^2 x} = \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}$$

$$= \cot x \csc x$$

$$= \int -\csc x \cot x \,dx$$

$$= \csc x + C$$

Note:

$$\int_a^b f(x)\,dx$$

is called the "definite integral of f from a to b." And, it is a number.

$$\int f(x)\,dx$$

is called an "indefinite integral of f". And, it is a family of functions.