May 21 Math 2254 sec 001 Summer 2015

Section 5.5: The Indefinite Integral

New notation for antiderivatives:

If $F'(x) = f(x)$, i.e. F is any antiderivative of f, we will write

$$\int f(x)\,dx = F(x) + C$$

and we'll call $\int f(x)\,dx$ the **indefinite** integral of f.

For example:

$$\int 2x\,dx = x^2 + C, \quad \text{and} \quad \int \cos t\,dt = \sin t + C$$

Note:

$$\int_a^b f(x)\,dx$$

is called the "definite integral of f from a to b." And, it is a number.

$$\int f(x)\,dx$$

is called an "indefinite integral of f". And, it is a family of functions.

Application

When a living organism dies, the Carbon-14 present in it decays at a rate proportional to the amount present at the time of death. Letting the amount of Carbon-14 at time t be denoted by A, the differential equation

$$\frac{dA}{dt} = -kA$$

where k is a positive constant models this process. Solve this equation for a family of functions $A(t)$. Assume that when $t = 0$, the initial Carbon-14 mass is A_0.

derivative is negative hence A is decreasing

$$\frac{dA}{dt} = -kA$$

rate of change of A w/r/t time

proportional to A

$$\frac{dA}{dt} = -kA$$

multiply by dt

divide by A

$$\frac{dA}{dt}\,dt = -kA\,dt$$

$$\frac{1}{A}\frac{dA}{dt}\,dt = -k\,dt$$

$$\frac{1}{A}\,dA = -k\,dt$$

Integrate both sides

$$\int \frac{1}{A}\,dA = \int -k\,dt$$

$$\ln|A| = -kt + C$$

A can't be negative

exponentiate

$$e^{\ln A} = e^{-kt+C}$$

$$* \quad A = e^{C} e^{-kt}$$

use the condition $A(0) = A_0$

$$A_0 = e^{C} e^{-k \cdot 0}$$

$$= e^{C} \cdot 1 \Rightarrow e^{C} = A_0$$

plugging into *

$$\boxed{A(t) = A_0 e^{-kt}}$$

Half Life of Carbon-14

We found that $A(t) = A_0 e^{-kt}$. It is known that the half-life of Carbon-14 is 5730 years. For t in years, determine the value of k.

When $t = 5730$ years $A = \frac{1}{2}$ its original amount

$$A(5730) = \frac{1}{2}A_0 = A_0\, e^{-k(5730)}$$

$$\frac{1}{2} = e^{-k(5730)}$$ take log

$$\ln\frac{1}{2} = \ln\left(e^{-k(5730)}\right) = -k(5730)$$

$$k = \frac{-1}{5730} \ln\left(\frac{1}{2}\right)$$

$$= \frac{1}{5730} \ln\left(\frac{1}{2}\right)^{-1}$$

$$\boxed{k = \frac{\ln 2}{5730}}$$

Section 5.6: The Substitution Rule

Definition: Let f be a differentiable function of x. The variable

$$dx$$

is called a *differential*. It is an **independent** variable. Letting $y = f(x)$, the differential

$$dy$$

is a **dependent** variable defined by

$$dy = f'(x)dx.$$

$$dy = \frac{dy}{dx}\, dx$$

Examples:

$dy = f'(x)\,dx$

(a) Given $y = \sin^2(x)$, express dy in terms of dx.

$dy = 2\sin(x)\cos(x)\,dx$

(b) Given $u = x^2 + 2x$, express du in terms of dx.

$du = (2x+2)\,dx$

(c) Given $u = \frac{x}{3} + 1$, express du in terms of dx. $u = \frac{1}{3}x + 1$

$$du = \frac{1}{3}\, dx$$

(d) Given $v = \theta^8$, express dv in terms of $d\theta$.

$$dv = 8\theta^7\, d\theta$$

$$\int_0^1 2x(x^2+1)^2\,dx = \frac{7}{3}$$

Evaluate this by letting $u = x^2 + 1$.

Substitute every term in the integral with an appropriate expression in u.

$u = x^2+1$ so $du = 2x\,dx$

When $x = 0$
$u = 0^2+1 = 1$

$x = 1$
$u = 1^2+1 = 2$

$$\int_0^1 (x^2+1)^2\,2x\,dx = \int_1^2 (u)^2\,du$$

$$= \int_1^2 u^2\,du$$

$$= \frac{u^3}{3}\Bigg|_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

$$\int_0^1 2x(x^2+1)^{10}\,dx$$

Evaluate this by letting $u = x^2 + 1$.

Substitute as before

$u = x^2+1 \qquad du = 2x\,dx$

when $x=0$, $u = 0^2+1 = 1$

$x = 1$, $u = 1^2+1 = 2$

$$\int_0^1 (x^2+1)^{10}\, 2x\,dx = \int_1^2 u^{10}\,du$$

$$= \frac{u^{11}}{11}\Big|_1^2 = \frac{2^{11}}{11} - \frac{1^{11}}{11} = \frac{2048-1}{11}$$

$$= \frac{2047}{11}$$

The Method of Substitution

Theorem: Suppose $u = g(x)$ is a differentiable function, and f is continuous on the range of g. Then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du.$$

This is often refered to as u**-substitution.**
This is the Chain Rule in reverse!

$$u = g(x) \quad \Rightarrow \quad du = g'(x)\, dx$$

Evaluate each Indefinite integral using Substitution as Needed

(a) $\displaystyle\int (3x+2)^3\,dx$

Let $u = 3x+2$

$du = 3\,dx$

$\Rightarrow \frac{1}{3}du = dx$

$= \frac{1}{3}\int (3x+2)^3\, 3\,dx$

$= \frac{1}{3}\int u^3\,du$

$= \frac{1}{3}\frac{u^4}{4} + C = \frac{u^4}{12} + C$

$= \frac{1}{12}(3x+2)^4 + C$

(b) $\int t \sec^2(t^2)\, dt$

Let $u = t^2$

$du = 2t\, dt$

$\frac{1}{2} du = t\, dt$

$= \int \sec^2(t^2)\, t\, dt$

$= \int \sec^2 u \cdot \frac{1}{2} \cdot du = \frac{1}{2} \int \sec^2 u\, du$

$= \frac{1}{2} \tan u + C$

$= \frac{1}{2} \tan t^2 + C$

(c) $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

Let $u = \sqrt{x} = x^{1/2}$

$du = \frac{1}{2} x^{-1/2}\,dx$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

$= \int (\cos\sqrt{x}) \frac{1}{\sqrt{x}}\,dx$

$= \int \cos u\,(2)\,du$

$= 2\int \cos u\,du = 2\sin u + C$

$= 2\sin\sqrt{x} + C$

A Subtle use of Substitution

Evaluate

$$\int x\sqrt{x+1}\,dx \quad \text{by taking} \quad u = x+1$$

$$du = dx$$

$$u = x+1 \Rightarrow x = u-1$$

$$\int x\sqrt{x+1}\,dx = \int (u-1)\sqrt{u}\,du$$

$$= \int (u-1)\,u^{1/2}\,du$$

$$= \int \left(u^{3/2} - u^{1/2}\right) du$$

$$= \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{5} (x+1)^{5/2} - \frac{2}{3} (x+1)^{3/2} + C$$

A Subtle use of Substitution

Evaluate

$$\int x\sqrt{x+1}\,dx \quad \text{by taking} \quad u = \sqrt{x+1}$$

Note that $u^2 = x + 1$.

$u^2 = x+1 \Rightarrow x = u^2 - 1$

$dx = 2u\,du$

$$\int x\sqrt{x+1}\,dx = \int (u^2-1)u(2u)\,du$$

$$= 2\int (u^2-1)u^2\,du$$

$$= 2\int (u^4 - u^2)\, du$$

$$= 2\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C$$

$$= \frac{2}{5}u^5 - \frac{2}{3}u^3 + C$$

$$= \frac{2}{5}\left(\sqrt{x+1}\right)^5 - \frac{2}{3}\left(\sqrt{x+1}\right)^3 + C$$

$$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$$

Some New Antiderivative rules

Use substitution to show that $\int \tan x \, dx = \ln|\sec x| + C$.

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

Let $u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$$= -\int \frac{1}{u} \, du$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

$$= \ln|(\cos x)^{-1}| + C = \ln|\sec x| + C$$

Some New Antiderivative rules

Use substitution to show that $\int \sec x\, dx = \ln|\sec x + \tan x| + C$.

$$\int \sec x\, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

Let $u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)\, dx$

$= (\sec^2 x + \sec x \tan x)\, dx$

$$= \int \frac{1}{u}\, du$$

$$= \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

Some New Antiderivative rules

$$\int \tan x \, dx = \ln|\sec x| + C, \quad \int \cot x \, dx = \ln|\sin x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C,$$

$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C$$

Theorem (Substitution for Definite Integrals)

Suppose g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$. Then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Evaluate each Definite Integral

(a) $\displaystyle\int_0^1 x\sqrt{1-x^2}\,dx$

$u = 1 - x^2$

$du = -2x\, dx$

$-\frac{1}{2}\, du = x\, dx$

When $x=0$, $u = 1-0^2 = 1$

$x=1$, $u = 1-1^2 = 0$

$$= \int_1^0 -\frac{1}{2}\sqrt{u}\; du$$

$$= -\frac{1}{2}\int_1^0 u^{1/2}\, du = -\frac{1}{2}\left.\frac{u^{3/2}}{3/2}\right|_1^0$$

$$= -\frac{1}{2}\cdot\frac{2}{3}\, u^{3/2}\Big|_1^0 = -\frac{1}{3}\, 0^{3/2} - \left(-\frac{1}{3}\cdot 1^{3/2}\right) = \frac{1}{3}$$