May 22 Math 2254 sec 001 Summer 2015

Section 5.6: The Substitution Rule

Theorem: (The Method of Substitution) Suppose $u = g(x)$ is a differentiable function, and f is continuous on the range of g. Then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du.$$

This is often refered to as u-**substitution.**
This is the Chain Rule in reverse!

Theorem: (Definite Integrals) Suppose g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$. Then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Evaluate each Definite Integral

(b) $\int_0^{\pi/4} \cos\left(2t - \frac{\pi}{4}\right)\, dt$

Let $u = 2t - \frac{\pi}{4}$

$du = 2\, dt$

$\frac{1}{2}\, du = dt$

when $t = 0$,

$u = 2 \cdot 0 - \frac{\pi}{4} = \frac{-\pi}{4}$

$t = \frac{\pi}{4}$

$u = 2 \cdot \frac{\pi}{4} - \frac{\pi}{4} = \frac{\pi}{4}$

$= \int_{-\frac{\pi}{4}}^{\pi/4} \cos(u) \cdot \frac{1}{2}\, du$

$= \frac{1}{2} \sin(u) \Big|_{-\pi/4}^{\pi/4}$

$= \frac{1}{2} \sin\left(\frac{\pi}{4}\right) - \frac{1}{2} \sin\left(\frac{-\pi}{4}\right)$

$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2}\left(\frac{-1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$

Options for Evaluating Definite Integrals with Substitution

Evaluate the definite integral in two ways.

(1) Use substitution for the entire definite integral including the limits.

(2) Find an anti-derivative using substitution, revert back to the original variable, and use the original limits.

$$\int_0^2 \frac{x}{x^2+3}\,dx$$

$$= \int_3^7 \frac{1}{2}\,\frac{du}{u}$$

(1) Let $u = x^2 + 3$

$du = 2x\,dx \Rightarrow \frac{1}{2}\,du = x\,dx$

when $x=0$, $u = 0^2+3 = 3$

$x = 2$, $u = 2^2+3 = 7$

$$= \frac{1}{2}\ln|u| \Big|_3^7 = \frac{1}{2}\ln|7| - \frac{1}{2}\ln|3|$$

(2) $$\int_0^2 \frac{x}{x^2+3}\,dx$$

Find $$\int \frac{x}{x^2+3}\,dx$$

Let $u = x^2+3$

$du = 2x\,dx$

$\frac{1}{2}\,du = x\,dx$

$$= \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|w| + C$$

$$= \frac{1}{2}\ln|x^2+3| + C$$

$$\int_0^2 \frac{x}{x^2+3}\,dx = \frac{1}{2}\ln|x^2+3|\Big|_0^2$$

$$= \frac{1}{2}\ln|2^2+3| - \frac{1}{2}\ln|0^2+3|$$

$$= \frac{1}{2}\ln|7| - \frac{1}{2}\ln|3|$$

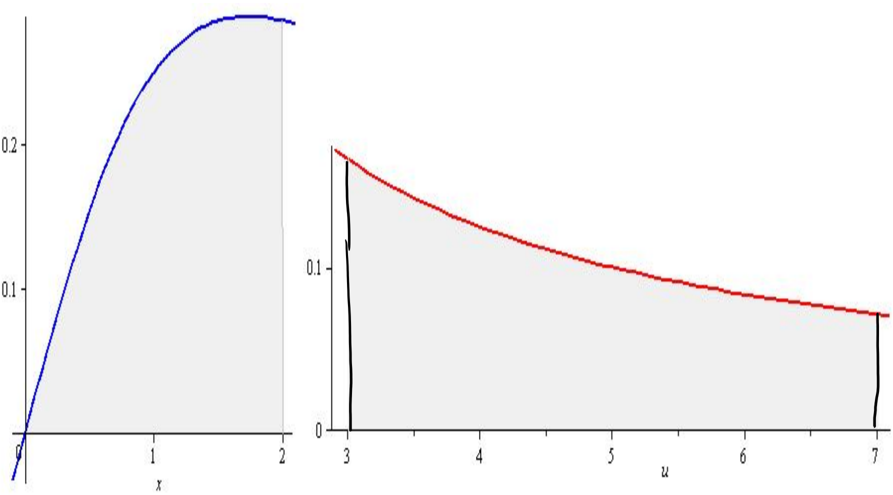

Figure: Equivalent Areas $\int_0^2 \frac{x}{x^2+3}\,dx = \int_3^7 \frac{1}{2}\frac{du}{u}$. The curve on the left is $f(x) = \frac{x}{x^2+3}$. The curve on the right is $g(u) = \frac{1}{2u}$.

The Substitution $u = ax$ for constant a

Evaluate $\int \cos(ax)\,dx$ where a is a nonzero constant.

let $u = ax \qquad du = a\,dx$

$$\Rightarrow \frac{1}{a}\,du = dx$$

$$\int \cos(ax)\,dx = \int \frac{1}{a}\cos(u)\,du$$

$$= \frac{1}{a}\sin(u) + C$$

$$= \frac{1}{a}\sin(ax) + C$$

The Substitution $u = ax$ for constant a

Evaluate $\int e^{ax}\,dx$ where a is a nonzero constant.

Let $u = ax$ $\quad du = a\,dx$

$\frac{1}{a}\,du = dx$

$$\int e^{ax}\,dx = \int \frac{1}{a} e^{u}\,du$$

$$= \frac{1}{a} e^{u} + C$$

$$= \frac{1}{a} e^{ax} + C$$

The Substitution $u = ax + b$ for constant a

Evaluate $\int \frac{1}{ax+b}\,dx$ where a is a nonzero constant and b is any constant.

Let $u = ax+b$ $\qquad du = a\,dx$

$\frac{1}{a}\,du = dx$

$$\int \frac{dx}{ax+b} = \int \frac{1}{a}\,\frac{du}{u}$$

$$= \frac{1}{a}\ln|u| + C$$

$$= \frac{1}{a}\ln|ax+b| + C$$

The Substitution $u = ax$ for constant a

We can generalize as follows.

$$\int f(ax)\,dx = \frac{1}{a}\int f(u)\,du$$

Some examples of this are

$$\begin{aligned}
\int \cos(ax)\,dx &= \tfrac{1}{a}\sin(ax) + C\,, & \int e^{ax}\,dx &= \tfrac{1}{a}e^{ax} + C\\
\int \sin(ax)\,dx &= -\tfrac{1}{a}\cos(ax) + C\,, & \int \tfrac{1}{ax+b}\,dx &= \tfrac{1}{a}\ln|ax+b| + C\\
\int \sec^2(ax)\,dx &= \tfrac{1}{a}\tan(ax) + C\,, & \int \tan(ax)\,dx &= \tfrac{1}{a}\ln|\sec(ax)| + C
\end{aligned}$$

Symmetry and Integrals from $-a$ to a

Recall: A function f is even if $f(-x) = f(x)$.
A function f is odd if $f(-x) = -f(x)$.

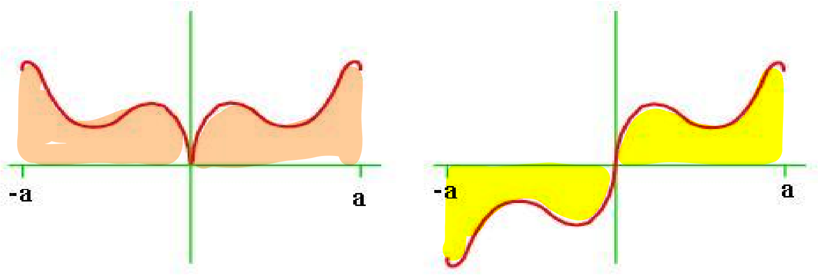

Figure: Symmetric Functions: Left is even, right is odd.

Theorem:

If f is an even, integrable function, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

If f is an odd, integrable function, then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Evaluate

$$\int_{-2.3}^{2.3} x^7 \cos(4x)\, dx$$

$f(x) = x^7 \cos(4x)$, $f(-x) = (-x)^7 \cos(-4x)$

$= -x^7 \cos(4x) = -f(x)$

f is odd

$$\int_{-2.3}^{2.3} x^7 \cos(4x)\, dx = 0$$ by symmetry.

Evaluate the Integral. Use Symmetry to Simplify the Process.

$$\int_{-2}^{2}(x^4+x^3+x^2)\,dx = \underbrace{\int_{-2}^{2}(x^4+x^2)\,dx}_{\text{even}} + \underbrace{\int_{-2}^{2}x^3\,dx}_{\text{odd}}$$

$$= 2\int_{0}^{2}(x^4+x^2)\,dx + 0$$

$$= 2\left(\frac{x^5}{5}+\frac{x^3}{3}\right)\Bigg|_0^2$$

$$= 2\left(\frac{2^5}{5} + \frac{2^3}{3}\right) - 2(0)$$

$$= 2\cdot 2^3\left(\frac{4}{5} + \frac{1}{3}\right) = 16\left(\frac{12+5}{15}\right) = \frac{16\cdot 17}{15}$$

$$= \frac{272}{15}$$

Section 6.1: Area Between Curves

Consider a pair of continuous curves $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$.

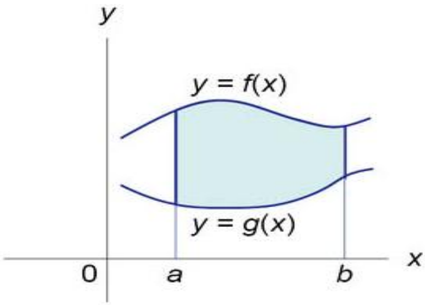

Figure: The curves enclose a region. We can ask what its area is.

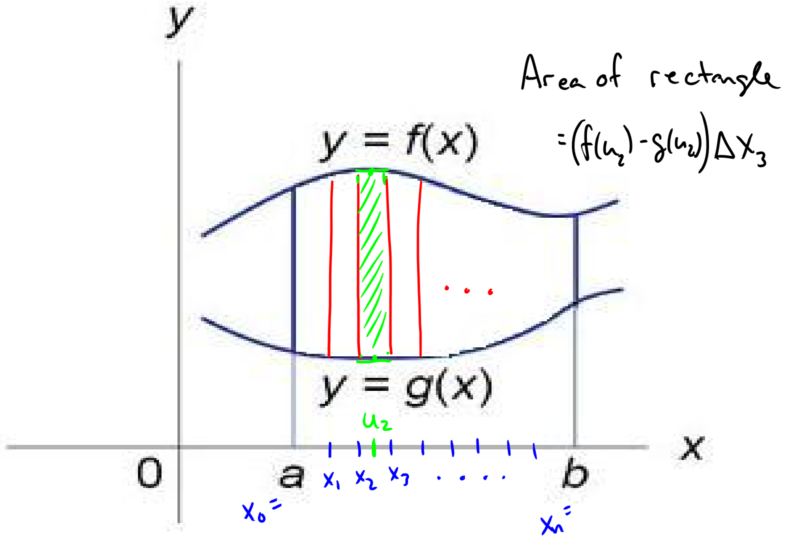

Figure: We can "build" the area from approximating rectangles.

Total area

$$A \approx \left(f(u_1) - g(u_1)\right)\Delta x_1 + \left(f(u_2) - g(u_2)\right)\Delta x_2$$

$$+ \dots + \left(f(u_n) - g(u_n)\right)\Delta x_n$$

$$\approx \sum_{i=1}^{n} \left(f(u_i) - g(u_i)\right)\Delta x_i$$

Riemann Sum

Area Between Curves:

Suppose f and g are continuous on $[a, b]$ and $f(x) \geq g(x)$. The area A bounded between the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is

$$A = \int_a^b (f(x) - g(x))\, dx.$$

$$= \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$