Convolution & Dirac Delta

The convolution operation is an extremely important operator in analysis and engineering. In EE, the main application is linear time invariant theory (LTI system theory), where the input-output behavior of an LTI system is described by an impulse response (next time!).

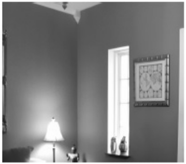

Figure: Original pic

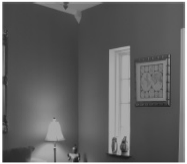

Figure: Sharpened pic

A 2-D convolution (we'll do 1-D) is used in image processing to perform edge detection and blurring. Deconvolution (a similar process) is used to sharpen images.

Convolution

Definition

If f and g are continuous, the **convolution** of f and g, denoted $f * g$ is defined for $t \geq 0$ by

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)\, d\tau.$$

Remark 1: Note that $f * g$ is a special type of **product** that produces a new function of t.

Remark 2: You might have noticed an integral of this form on the table of Laplace transforms. The Laplace transform of a convolution is related to a product of Laplace transforms.

Example

For $f(t) = e^t$ and $g(t) = e^{2t}$, evaluate $f * g$.

Theorem

Theorem

Suppose f and g are continuous on $[0, \infty)$ and of exponential order c for some $c \geq 0$, then $f * g$ has a Laplace transform. Moreover, for $s > c$,

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g).$$

Equivalently, if $F(s) = \mathcal{L}(f)$ and $G(s) = \mathcal{L}(g)$, then

$$\mathcal{L}^{-1}(F(s)G(s)) = (f * g)(t).$$

Remark: Recall that the transform of a product is **NOT** the product of the transforms (interals don't work that way)! This theorm provides a correct treatment of the inverse Laplace transform of a product.

Example

Evaluate $\mathscr{L}^{-1}\left(\frac{1}{(s-1)(s-2)}\right)$.

Example

Let's solve the mass-spring oscillator <u>for all forcing functions!!</u>

$$y'' + y = f(t), \quad y(0) = y'(0) = 0.$$

Theorem

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Let f be piecewise continuous and of exponential order c for some $c \geq 0$. Then

$$\mathcal{L}(-tf(t)) = F'(s), \quad s > c.$$

More generally,

$$\mathcal{L}(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} F(s).$$

Remark: Note that the first statement can be rearranged to produce

$$f(t) = \mathcal{L}^{-1}(F(s)) = -\frac{1}{t}\mathcal{L}^{-1}(F'(s)).$$

Example

Evaluate $\mathcal{L}(t^2 \sin(kt))$.

Theorem

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Let f be piecewise continuous and of exponential order c for some $c \geq 0$. Further suppose that $\lim_{t \to 0^+} \frac{f(t)}{t} < \infty$. Then

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\sigma)\, d\sigma, \quad s > c.$$

Remark: This can be rearranged to produce

$$f(t) = \mathcal{L}^{-1}(F(s)) = t\mathcal{L}^{-1}\left(\int_s^\infty F(\sigma)\, d\sigma\right).$$

Example

The expression $\dfrac{\sin(t)}{t}$ for $t \neq 0$ pops up in signal processing. It's referred to as a *sinc* function or cardinal sine. Evaluate $\mathcal{L}\left(\dfrac{\sin(t)}{t}\right)$.

Delta Functions

Consider a force, $f(t)$, that acts over a very short time interval $a \leq t \leq b$, with $f = 0$ outside of $[a, b]$.

Figure: Think of something like a lightning strike.

A quantity of interest is the **impulse**, $p = \int_a^b f(t)\,dt$.

Delta Functions

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Question: How should we model this?

We replace the force $f(t)$ with a *simple* force that has the same impulse.

Delta Functions

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$$d_{a,\epsilon}(t) = \begin{cases} \dfrac{1}{\epsilon}, & a \leq t < a + \epsilon \\ 0, & \text{otherwise} \end{cases}$$

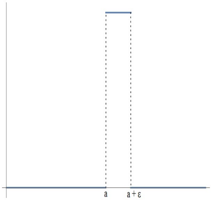

Figure: Note that the value of ϵ determines the width and height of the rectangle, but the area enclosed is always 1.

Delta Functions

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Now think of an instantaneous impulse precisely at $t = a$. We model this by letting $\epsilon \to 0$ and defining

$$\delta_a(t) = \lim_{\epsilon \to 0} d_{a,\epsilon}(t).$$

If we could take the limit of (1), and move the limit inside the integral, we would get

$$\int_a^\infty \delta_a(t)\,dt = 1.$$

Delta Functions

Trying to look at δ_a pointwise, we get

$$\delta_a(t) = \begin{cases} +\infty, & t = a \\ 0, & t \neq a \end{cases}$$

So δ_a isn't really a function[1]. We call δ_a a **Dirac delta function**.

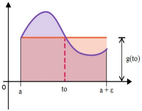
It is defined by how it acts on functions when combined with integration.

[1] It's something that is called a *generalized function* or sometimes a *functional* or a *distribution*.

Defining the Dirac Delta

Suppose g is continuous and $\epsilon > 0$. The MVT says that there exists some t_0 in $[a, a + \epsilon]$, such that

$$\int_a^{a+\epsilon} g(t)\,dt = \epsilon g(t_0).$$



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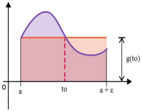

Using continuity and taking the limit as $\epsilon \to 0$, note that

$$\lim_{\epsilon\to 0}\int_0^{\infty} g(t)d_{a,\epsilon}(t)\,dt = \lim_{\epsilon\to 0}\int_a^{a+\epsilon}\frac{g(t)}{\epsilon}\,dt = \lim_{\epsilon\to 0} g(t_0) = g(a).$$

We'll use this result to define δ_a.

Dirac Delta

Definition

For each continuous function g, define $\delta_a(t)$ via

$$\int_0^\infty g(t)\delta_a(t)\,dt = g(a).$$

Remark: This property of the Dirac delta *function* is referred to as a *sifting property*.

The Laplace Transform of δ_a

From the previous definition, we have

$$\int_0^\infty e^{-st}\delta_a(t)\,dt = e^{-as}.$$

Hence

$$\mathcal{L}(\delta_a(t)) = e^{-as}, \quad a \geq 0.$$

Remark: We can use the Laplace transform to solve differential equations with Dirac delta forcing.

Example

Consider a object released from rest with initial position $y(0) = 3$. Assume that the ratio of spring constant to mass is $\frac{k}{m} = 4$ and that damping is negligible. At time $t = 2\pi$, the object is struck with a hammer, providing an impulse $p = 8$. Determine the position of the object for all $t > 0$ (i.e., find $y(t)$).